MANIFOLD STRUCTURE OF SPACES OF SPHERICAL TIGHT FRAMES

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Abstract. We consider the space $\mathcal{F}^{\mathbf{E}}_{k,n}$ of all spherical tight frames of k vectors in the n–dimensional Hilbert space $\mathbf{E}^n$ ($k > n$), for $\mathbf{E} = \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$, and its orbit space $\mathcal{G}^{\mathbf{E}}_{k,n} = \mathcal{F}^{\mathbf{E}}_{k,n}/\mathcal{O}^{\mathbf{E}}_n$ under the obvious action of the group $\mathcal{O}^{\mathbf{E}}_n$ of structure preserving transformations of $\mathbf{E}^n$. We show that the quotient map $\mathcal{F}^{\mathbf{E}}_{k,n} \to \mathcal{G}^{\mathbf{E}}_{k,n}$ is a locally trivial fiber bundle (also in the more general case of ellipsoidal tight frames) and that there is a homeomorphism $\mathcal{G}^{\mathbf{E}}_{k,n} \to \mathcal{G}^{\mathbf{E}}_{k,k-n}$. We show that $\mathcal{G}^{\mathbf{E}}_{k,n}$ and $\mathcal{F}^{\mathbf{E}}_{k,n}$ are real manifolds whenever k and n are relatively prime, and we describe them as a disjoint union of finitely many manifolds (of various dimensions) when when k and n have a common divisor. We also prove that $\mathcal{F}^{\mathbf{R}}_{k,2}$ is connected ($k \geq 4$) and $\mathcal{F}^{\mathbf{R}}_{n+2,n}$ is connected, ($n \geq 2$). The spaces $\mathcal{G}^{\mathbf{R}}_{4,2}$ and $\mathcal{G}^{\mathbf{R}}_{5,2}$ are investigated in detail. The former is found to be a graph and the latter is the orientable surface of genus 25.

1. Introduction

A *frame* is a list of vectors $F = (f_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$ satisfying

$$A\|v\|^2 \leq \sum_{i\in I} |\langle v, f_i\rangle|^2 \leq B\|v\|^2 \qquad (v \in \mathcal{H}) \tag{1}$$

for some constants $0 < A \leq B$; the optimal such constants are called the *frame bounds* of F. The frame F is finite if the index set I is finite, which implies $\mathcal{H}$ is finite dimensional. An example is an orthonormal basis; however, in general a frame may have redundancies, and these are essential in many recent applications of frames (including finite frames) to signal processing — see [7], [2] and references cited by these. The frame F is said to be *tight* if the constants A and B in (1) can be taken to be equal to each other. Some recent references on finite frame theory are [1]–[8] and [10].

In this paper, we will consider finite frames, in both the real and complex cases, i.e. $\mathcal{H} = \mathbf{E}^n$ for $\mathbf{E} = \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$. We will be primarily interested in frames all of whose vectors f_i lie on the unit sphere of $\mathbf{E}^n$, i.e. the *spherical tight frames*. (These are also called equal–norm tight frames, uniform tight frames and normalized tight frames in the literature.) Our focus will be on the set $\mathcal{F}^{\mathbf{E}}_{k,n}$ of all spherical tight frames of k vectors in $\mathbf{E}^n$, for $k > n$, and in particular on the topological questions of connectedness and the manifold structure of $\mathcal{F}^{\mathbf{E}}_{k,n}$.

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The technical key to our results is to consider the orbit space $\mathcal{G}^{\mathbf{E}}_{k,n} = \mathcal{F}^{\mathbf{E}}_{k,n}/\mathcal{O}^{\mathbf{E}}_n$ for the obvious action of the group of inner–product preserving transformations $\mathcal{O}^{\mathbf{E}}_n$ of the Hilbert space $\mathbf{E}^n$. (Thus, $\mathcal{O}^{\mathbf{R}}_n$ is the group of $n \times n$ orthogonal matrices, and $\mathcal{O}^{\mathbf{C}}_n$ is the group of $n \times n$ unitary matrices.) It is well known (*cf* [2] and [8]) that $\mathcal{G}^{\mathbf{E}}_{k,n}$ can be naturally identified with the subset of the Grassman manifold of n–planes in $\mathbf{E}^k$ consisting of projections all of whose diagonal entries are equal to n/k. We observe that the quotient map $\mathcal{F}^{\mathbf{E}}_{k,n} \to \mathcal{G}^{\mathbf{E}}_{k,n}$ is a locally trivial fiber bundle (with fibers $\mathcal{O}^{\mathbf{E}}_n$). In fact, we treat a more general case of ellipsoidal tight frames — see §2. An important consequence is that $\mathcal{G}^{\mathbf{E}}_{k,n}$ and $\mathcal{G}^{\mathbf{E}}_{k,k-n}$ are homeomorphic.

For $n \geq 2$, since $\mathcal{F}^{\mathbf{E}}_{n+1,1}$ and hence also $\mathcal{G}^{\mathbf{E}}_{n+1,1}$ is easy to describe, the homeomorphism $\mathcal{G}^{\mathbf{E}}_{n+1,1} \to \mathcal{G}^{\mathbf{E}}_{n+1,n}$ allows us to analyze the space $\mathcal{F}^{\mathbf{E}}_{n+1,n}$ of all spherical tight frames of $n+1$ vectors in $\mathbf{E}^n$. In the real case, we thereby reprove the result [7] that all such frames are equivalent to each other if one allows orthogonal transformations of $\mathbf{R}^n$ and negating some vectors; we also prove the analogous result in the complex case. Finally, we use these techniques to write down explicitly a prototypical example of a spherical tight frame of $n+1$ vectors in $\mathbf{R}^n$, from which all frames in $\mathcal{F}^{\mathbf{R}}_{n+1,n}$ and $\mathcal{F}^{\mathbf{C}}_{n+1,n}$ can be obtained.

Both $\mathcal{F}^{\mathbf{E}}_{k,n}$ and $\mathcal{G}^{\mathbf{E}}_{k,n}$ are real algebraic sets. By classical results of Whitney [11], each of these can, therefore, be written as a disjoint union of finitely many manifolds. We explicitly describe such a decomposition. When k and n are relatively prime, we show that $\mathcal{G}^{\mathbf{E}}_{k,n}$ is itself a real analytic manifold, and, therefore, so is $\mathcal{F}^{\mathbf{E}}_{k,n}$. When n and k are not relatively prime, $\mathcal{G}^{\mathbf{E}}_{k,n}$ is written as a disjoint union of manifolds, corresponding to block diagonal decompositions of projections. We get a similar description of $\mathcal{F}^{\mathbf{E}}_{k,n}$. In particular, we say a tight frame $F = (f_1, \ldots, f_k)$ for $\mathbf{E}^n$ is *orthodecomposable* if the vectors in F can be partitioned into proper sublists which form tight frames for orthogonal subspaces of $\mathbf{E}^n$. (See Definition 4.8.) Let $\hat{M}^{\mathbf{E}}_{k,n}$ be the set of spherical tight frames in $\mathcal{F}^{\mathbf{E}}_{k,n}$ that are not orthodecomposable. Then $\hat{M}^{\mathbf{E}}_{k,n}$ is a nonempty manifold, and $\mathcal{F}^{\mathbf{E}}_{k,n}$ is the union of $\hat{M}^{\mathbf{E}}_{k,n}$ together with other manifolds (of lower dimension) corresponding to orthodecomposability according to certain partitions.

Another consequence of Whitney's results [11] is that $\mathcal{F}^{\mathbf{E}}_{k,n}$ and $\mathcal{G}^{\mathbf{E}}_{k,n}$ have only finitely many connected components. By considering the rearrangement of chains in $\mathbf{R}^2$, we prove that the space $\mathcal{F}^{\mathbf{R}}_{k,2}$ of tight spherical frames of k vectors in $\mathbf{R}^2$ is connected for all $k \geq 4$, and from this result we obtain that the space $\mathcal{F}^{\mathbf{R}}_{n+2,n}$ of real tight spherical frames with two redundant vectors is connected, for all $n \geq 2$.

About half of the length of this paper is occupied with detailed consideration of two examples: $\mathcal{G}^{\mathbf{R}}_{4,2}$ and $\mathcal{G}^{\mathbf{R}}_{5,2}$ (the latter of which is homeomorphic to $\mathcal{G}^{\mathbf{R}}_{5,3}$). We find that $\mathcal{G}^{\mathbf{R}}_{4,2}$ is a graph with twelve vertices and twenty–four edges, and $\mathcal{G}^{\mathbf{R}}_{5,2}$ is the orientable surface of genus 25. Similar techniques should permit the description of $\mathcal{G}^{\mathbf{R}}_{k,2}$ for larger k, though with considerably more work. These examples inspired our results on the manifold structure of $\mathcal{G}^{\mathbf{E}}_{k,n}$ for general k and n.

The organization of this paper is as follows. In §2, we show that the quotient map $\mathcal{F}^{\mathbf{E}}_{k,n} \to \mathcal{G}^{\mathbf{E}}_{k,n}$ is a locally trivial fiber bundle, and the analogous result for ellipsoidal

tight frames. In §3, we describe $\mathcal{G}^{\mathbf{E}}_{n+1,n}$ and give a concrete example of $F \in \mathcal{F}^{\mathbf{R}}_{n,n+1}$. In §4, we prove $\mathcal{G}^{\mathbf{E}}_{k,n}$ and $\mathcal{F}^{\mathbf{E}}_{k,n}$ are manifolds when k and n are relatively prime, and more generally, we write any $\mathcal{G}^{\mathbf{E}}_{k,n}$ and $\mathcal{F}^{\mathbf{E}}_{k,n}$ as a disjoint union of finitely many manifolds. In §5, we elucidate $\mathcal{G}^{\mathbf{R}}_{4,2}$, and in §6 we show $\mathcal{G}^{\mathbf{R}}_{5,2}$ is the orientable surface of genus 25. In §7, we show $\mathcal{F}^{\mathbf{E}}_{k,n}$ is connected if and only if $\mathcal{F}^{\mathbf{E}}_{k,k-n}$ is connected, and we prove that $\mathcal{F}^{\mathbf{R}}_{k,2}$ (for $k \geq 4$) and $\mathcal{F}^{\mathbf{R}}_{n+2,2}$ (for $n \geq 2$) are connected.

2. Equivalence classes of ellipsoidal tight frames

Let $F = (f_1, \ldots, f_k)$ be an ordered frame of k vectors in $\mathbf{E}^n$, where $\mathbf{E} = \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$. Associated to F is its *synthesis operator* $\mathbf{E}^k \to \mathbf{E}^n$, defined by

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \mapsto \sum_{j=1}^{k} c_j f_j. \tag{2}$$

The matrix of this operator is thus the $n \times k$ matrix whose columns are the vectors $f_1, \ldots, f_k$ in this order, and we will identify F with this matrix; thus we also use the notation F for the synthesis operator (2) itself. The *analysis operator* is the adjoint $F^* : \mathbf{E}^n \to \mathbf{E}^k$, given by

$$F^*(v) = \begin{pmatrix} \langle v, f_1 \rangle \\ \vdots \\ \langle v, f_k \rangle \end{pmatrix}.$$

Suppose F is a tight frame with frame bound B. Then $B^{-1/2} F^* : \mathbf{E}^n \to \mathbf{E}^k$ is an isometry. By a dimensionality argument, there is $U \in \mathcal{O}^{\mathbf{E}}_k$ such that

$$F = B^{1/2} W_{n,k} U, \tag{3}$$

where $W_{n,k} = (I_n | 0_{n,k-n})$ is the $n \times k$ matrix having 1 in each (i,i)th position and zeros elsewhere. Conversely, whenever $S : \mathbf{E}^n \to \mathbf{E}^k$ is an isometry, $F = B^{1/2} S^*$ is a tight frame of k vectors in $\mathbf{E}^n$ having frame bound B.

Let $a = (a_1, \ldots, a_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and consider the ellipsoid

$$\mathcal{E}^{\mathbf{E}}(a) = \{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbf{E}^n \mid \sum_{j=1}^{n} a_j |v_j|^2 = 1 \}.$$

Letting

$$D_n(a) = \operatorname{diag}(a_1, \ldots, a_n) \in M_n(\mathbf{R}),$$

we have

$$\mathcal{E}^{\mathbf{E}}(a) = \{ v \in \mathbf{E}^n \mid \langle D_n(a) v, v \rangle = 1 \}. \tag{4}$$

Let $\mathcal{F}^{\mathbf{E}}_k(a)$ denote the set of all ordered tight frames of k vectors that lie on the ellipsoid $\mathcal{E}^{\mathbf{E}}(a)$. These are the *ellipsoidal tight frames* (ETFs) of k vectors on $\mathcal{E}^{\mathbf{E}}(a) \subseteq \mathbf{E}^n$. An elementary construction was given in [6] showing that $\mathcal{F}^{\mathbf{E}}_k(a)$ is always nonempty. Let $\mathcal{O}^{\mathbf{E}}_n$ act in the usual way on $\mathbf{E}^n$ by left multiplication and let

$$\mathcal{T}^{\mathbf{E}}_n(a) = \{ V \in \mathcal{O}^{\mathbf{E}}_n \mid V(\mathcal{E}^{\mathbf{E}}(a)) = \mathcal{E}^{\mathbf{E}}(a) \}$$

be the subgroup of those elements of $\mathcal{O}^{\mathbf{E}}_n$ that preserve $\mathcal{E}^{\mathbf{E}}(a)$. From (4), we get

$$\mathcal{T}^{\mathbf{E}}_n(a) = \{ U \in \mathcal{O}^{\mathbf{E}}_n \mid U^* D_n(a) U = D_n(a) \}. \tag{5}$$

Then $\mathcal{T}_n^{\mathbf{E}}(a)$ acts on $\mathcal{F}_k^{\mathbf{E}}(a)$ by left multiplication, where a frame $F \in \mathcal{F}_k^{\mathbf{E}}(a)$ is represented as the $n \times k$ matrix of its synthesis operator, as described above. Since the rank of every $F \in \mathcal{F}_k^{\mathbf{E}}(a)$ is n, this action is free. We will study the space of orbits of this action.

Let $\pi : \mathcal{F}_k^{\mathbf{E}}(a) \to M_n(\mathbf{E})$ be defined by $\pi(F) = F^* D_n(a) F$. Since the frame $F \in \mathcal{F}_k^{\mathbf{E}}(a)$ consists of vectors lying on the ellipsoid $\mathcal{E}^{\mathbf{E}}(a)$, by (4) each diagonal entry of $\pi(F)$ is equal to 1. Thus $\mathrm{Tr}(\pi(F)) = k$. On the other hand, letting $U \in \mathcal{O}_k^{\mathbf{E}}$ be such that $F = B^{1/2} W_{n,k} U$, we have

$$\pi(F) = BU^* W_{n,k}^* D_n(a) W_{n,k} U = BU^* D_k(a) U, \tag{6}$$

where

$$D_k(a) = \mathrm{diag}(a_1, \ldots, a_n, 0, \ldots, 0) \in M_k(\mathbf{R}).$$

Hence $\mathrm{Tr}(\pi(F)) = B(a_1 + \cdots + a_n)$ and the frame bound for F is

$$B = k/(a_1 + \cdots + a_n). \tag{7}$$

Proposition 2.1. *Let $F, G \in \mathcal{F}_k^{\mathbf{E}}(a)$. Then F and G lie in the same $\mathcal{T}_n^{\mathbf{E}}(a)$–orbit if and only if $\pi(F) = \pi(G)$. Furthermore, the image of π is*

$$\mathcal{G}_k^{\mathbf{E}}(a) \overset{def}{=} \{R = \tfrac{k}{a_1 + \cdots + a_n} U^* D_k(a) U \mid U \in \mathcal{O}_k^{\mathbf{E}},\ R_{ii} = 1,\ (1 \le i \le k)\},$$

where R_{ii} is the ith diagonal entry of R.

Proof. If F and G lie in the same $\mathcal{T}_n^{\mathbf{E}}(a)$–orbit, then there is $U \in \mathcal{T}_n^{\mathbf{E}}(a)$ such that $G = UF$. From (5), we get $\pi(G) = F^* U^* D_n(a) U F = F^* D_n(a) F = \pi(F)$.

On the other hand, suppose $\pi(G) = \pi(F)$. Let $U, V \in \mathcal{O}_k^{\mathbf{E}}$ be such that $F = B^{1/2} W_{n,k} U$ and $G = B^{1/2} W_{n,k} V$. Using (6) we get $VU^* D_k(a) U V^* = D_k(a)$. Since $a_n > 0$, this yields

$$VU^* = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right),$$

where $X \in \mathcal{T}_n^{\mathbf{E}}(a)$ and $Y \in \mathcal{O}_{k-n}^{\mathbf{E}}$. Therefore,

$$G = B^{1/2} W_{n,k} \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right) U = B^{1/2} X W_{n,k} U = XF,$$

i.e. G lies in the same $\mathcal{T}_n^{\mathbf{E}}(a)$–orbit as F.

The inclusion $\pi(\mathcal{F}_k^{\mathbf{E}}(a)) \subseteq \mathcal{G}_k^{\mathbf{E}}(a)$ is demonstrated in the paragraph immediately preceding the lemma. If $R = \frac{k}{a_1 + \cdots + a_n} U^* D_k(a) U \in \mathcal{G}_k^{\mathbf{E}}(a)$ for $U \in \mathcal{O}_k^{\mathbf{E}}$, then letting $F = B^{1/2} W_{n,k} U$ with B as in (7), we have that F is a tight frame and $R = F^* D_n(a) F$. If f_i is the ith column of F, then $1 = R_{ii} = \langle D_n(a) f_i, f_i \rangle$, so $f_i \in \mathcal{E}^{\mathbf{E}}(a)$, and therefore $F \in \mathcal{F}_k^{\mathbf{E}}(a)$. □

Theorem 2.2. *The map $\pi : \mathcal{F}_k^{\mathbf{E}}(a) \to \mathcal{G}_k^{\mathbf{E}}(a)$ is a locally trivial fiber bundle with fiber $\mathcal{T}_n^{\mathbf{E}}(a)$.*

Proof. From Proposition 2.1 and the freeness of the $\mathcal{T}_n^{\mathbf{E}}(a)$–action, we have that π is surjective and, for every $R \in \mathcal{G}_k^{\mathbf{E}}(a)$, $\pi^{-1}(\{R\})$ is homeomorphic to $\mathcal{T}_n^{\mathbf{E}}(a)$. It remains to show local triviality. For this, it will suffice to find local sections of π, namely, given $R \in \mathcal{G}_k^{\mathbf{E}}(a)$ to find a neighborhood $\mathcal{U}$ of R and a continuous map $\mu : \mathcal{U} \to \mathcal{F}_k^{\mathbf{E}}(a)$ such that $\pi \circ \mu = \mathrm{id}_{\mathcal{U}}$, because then by Proposition 2.1, the map $\mathcal{T}_n^{\mathbf{E}}(a) \times \mathcal{U} \to \mathcal{F}_k^{\mathbf{E}}(a)$

given by $(U,S) \mapsto U\mu(S)$ will be a homeomorphism from $\mathcal{T}_n^{\mathbf{E}}(a) \times \mathcal{U}$ onto $\pi^{-1}(\mathcal{U})$ whose composition with π is the projection onto the second component $\mathcal{U}$.

Let

$$\mathcal{C}_k^{\mathbf{E}}(a) = \{U^* D_k(a) U \mid U \in \mathcal{O}_k^{\mathbf{E}}\}$$

and let

$$\sigma : \mathcal{O}_k^{\mathbf{E}} \to \mathcal{C}_k^{\mathbf{E}}(a)$$

be $\sigma(U) = U^* D_k(a) U$. Consider the closed subgroup

$$\mathcal{S}_k^{\mathbf{E}}(a) = \{U \in \mathcal{O}_k^{\mathbf{E}} \mid U D_k(a) = D_k(a) U\}$$

of $\mathcal{O}_k^{\mathbf{E}}$ and let $\mathcal{S}_k^{\mathbf{E}}(a)\backslash\mathcal{O}_k^{\mathbf{E}}$ denote the homogeneous space of right cosets of $\mathcal{S}_k^{\mathbf{E}}(a)$. The usual quotient map $q : \mathcal{O}_k^{\mathbf{E}} \to \mathcal{S}_k^{\mathbf{E}}(a)\backslash\mathcal{O}_k^{\mathbf{E}}$ is a locally trivial fiber bundle with fiber $\mathcal{S}_k^{\mathbf{E}}(a)$. The map $\mathcal{S}_k^{\mathbf{E}}(a)\backslash\mathcal{O}_k^{\mathbf{E}} \to \mathcal{C}_k^{\mathbf{E}}(a)$ given by $\mathcal{S}_k^{\mathbf{E}}(a)U \mapsto U^* D_k(a) U$ is a homeomorphism. Hence the map σ is a locally trivial fiber bundle with fiber $\mathcal{S}_k^{\mathbf{E}}(a)$.

Let

$$\widetilde{\mathcal{C}}_k^{\mathbf{E}}(a) = \{S \in \mathcal{C}_k^{\mathbf{E}}(a) \mid S_{ii} = \frac{a_1 + \cdots + a_n}{k}, \ (1 \le i \le k)\}$$

and let $\mathcal{V}_k^{\mathbf{E}}(a) = \sigma^{-1}(\widetilde{\mathcal{C}}_k^{\mathbf{E}}(a))$. The map $r : \widetilde{\mathcal{C}}_k^{\mathbf{E}}(a) \to \mathcal{G}_k^{\mathbf{E}}(a)$ of scalar multiplication by $\frac{k}{a_1+\cdots+a_n}$ is a surjective homeomorphism. From the proof of Proposition 2.1, if $U \in \mathcal{V}_k^{\mathbf{E}}(a)$, then letting $F = B^{-1/2} W_{n,k} U$ we have $F \in \mathcal{F}_k^{\mathbf{E}}(a)$; moreover, all elements of $\mathcal{F}_k^{\mathbf{E}}(a)$ arise in this way. Therefore, the map $\rho : \mathcal{V}_k^{\mathbf{E}}(a) \to \mathcal{F}_k^{\mathbf{E}}(a)$ defined by $\rho(U) = B^{-1/2} W_{n,k} U$ is surjective and continuous, and the diagram

$$\begin{tikzcd}
\mathcal{O}_k^{\mathbf{E}} \arrow[d, "\sigma"] & \mathcal{V}_k^{\mathbf{E}}(a) \arrow[l, hook'] \arrow[r, "\rho"] \arrow[d, "\sigma\restriction_{\mathcal{V}_k^{\mathbf{E}}(a)}"] & \mathcal{F}_k^{\mathbf{E}}(a) \arrow[d, "\pi"] \\
\mathcal{C}_k^{\mathbf{E}}(a) & \widetilde{\mathcal{C}}_k^{\mathbf{E}}(a) \arrow[l, hook'] \arrow[r, "r"] & \mathcal{G}_k^{\mathbf{E}}(a)
\end{tikzcd} \tag{8}$$

commutes. Suppose $R \in \mathcal{G}_k^{\mathbf{E}}(a)$ and let $R' = \frac{a_1+\cdots+a_n}{k} R = r^{-1}(R) \in \widetilde{\mathcal{C}}_k^{\mathbf{E}}(a)$. There is a neighborhood $\mathcal{U}'$ of R' in $\widetilde{\mathcal{C}}_k^{\mathbf{E}}(a)$ and a local section $\tau : \mathcal{U}' \to \mathcal{V}_k^{\mathbf{E}}(a)$ of σ, which is the restriction to $\widetilde{\mathcal{U}} \cap \widetilde{\mathcal{C}}_k^{\mathbf{E}}(a)$ of a local section $\tilde{\tau} : \widetilde{\mathcal{U}} \to \mathcal{O}_k^{\mathbf{E}}$, for some neighborhood $\widetilde{\mathcal{U}}$ of R in $\mathcal{C}_k^{\mathbf{E}}(a)$, satisfying $\sigma \circ \tau = \mathrm{id}_{\mathcal{U}'}$. Consider the neighborhood $\mathcal{U} = r(\mathcal{U}')$ of R in $\mathcal{G}_k^{\mathbf{E}}(a)$. Let

$$\mu = \rho \circ \tau \circ r^{-1}\restriction_{\mathcal{U}} : \mathcal{U} \to \mathcal{F}_k^{\mathbf{E}}(a).$$

Then $\pi \circ \mu = \mathrm{id}_{\mathcal{U}}$. Hence μ is a local section of π. $\square$

Remark 2.3. The local section $\tilde{\tau}$ can be taken to be real analytic.

A consequence of Proposition 2.1 and Theorem 2.2 is that $\mathcal{G}_k^{\mathbf{E}}(a)$, endowed with the relative topology from $M_k(\mathbf{E})$, is homeomorphic to the orbit space of the action of $\mathcal{T}_n^{\mathbf{E}}(a)$ on $\mathcal{F}_k^{\mathbf{E}}(a)$, endowed with the quotient topology.

Remark 2.4. If we wish to consider *unordered* frames, we should consider the action of the permutation group $\mathfrak{S}_k$ on $\mathcal{F}_k^{\mathbf{E}}(a)$ by

$$\mathcal{F}_k^{\mathbf{E}}(a) \times \mathfrak{S}_k \ni (F, \sigma) \mapsto (f_{\sigma(1)}, \ldots, f_{\sigma(k)}) = F A_\sigma,$$

where A_σ is the $k\times k$ permutation matrix associated to σ. Since this action commutes with the action of $\mathcal{T}_n^{\mathbf{E}}(a)$ on $\mathcal{F}_k^{\mathbf{E}}(a)$ it descends to the action

$$\mathcal{G}_n^{\mathbf{E}}(a)\times\mathfrak{S}_k \ni (R,\sigma)\mapsto A_\sigma^* R A_\sigma$$

of $\mathfrak{S}_k$ on $\mathcal{G}_n^{\mathbf{E}}(a)$, and $\mathcal{G}_n^{\mathbf{E}}(a)/\mathfrak{S}_k$ is the orbit space for the action of $\mathcal{T}^{\mathbf{E}}(a)$ on the set of unordered ellipsoidal tight frames.

Remark 2.5. Let $\mathcal{D}_k^{\mathbf{C}}=\mathbf{T}^k$ and $\mathcal{D}_k^{\mathbf{R}}=\mathcal{D}_k^{\mathbf{C}}\cap\mathbf{R}^k=\{\pm1\}^k$. If $\zeta=(\zeta_1,\ldots,\zeta_k)\in\mathcal{D}_k^{\mathbf{E}}$ and if $F=(f_1,\ldots,f_k)\in\mathcal{F}_k^{\mathbf{E}}(a)$, then setting

$$F\cdot\zeta=F\mathrm{diag}(\zeta_1,\ldots,\zeta_k)=(\zeta_1 f_1,\ldots,\zeta_k f_k),$$

we have $F\cdot\zeta\in\mathcal{F}_k^{\mathbf{E}}(a)$, and this defines an action of the multiplicative group $\mathcal{D}_k^{\mathbf{E}}$ on $\mathcal{F}_k^{\mathbf{E}}(a)$. Since this action commutes with the action of $\mathcal{T}_n^{\mathbf{E}}(a)$, it descends to the action of $\mathcal{D}_k^{\mathbf{E}}$ on $\mathcal{G}_k^{\mathbf{E}}(a)$ given by

$$\mathcal{G}_k^{\mathbf{E}}(a)\times\mathcal{D}_k^{\mathbf{E}}\ni(R,\zeta)\mapsto\mathrm{diag}(\overline{\zeta_1},\ldots,\overline{\zeta_k})R\,\mathrm{diag}(\zeta_1,\ldots,\zeta_k).$$

We now specialize to the case of spherical tight frames (STFs), namely when $a_1=\cdots=a_n=1$, which we will study in the remainder of the paper. The following corollary restates Proposition 2.1 and Theorem 2.2 in this case, and introduces the notation we will use. As usual, a *projection* in $M_k(\mathbf{C})$ or $M_k(\mathbf{R})$ is a self–adjoint idempotent.

Corollary 2.6. *Let $\mathcal{F}_{k,n}^{\mathbf{E}}$ denote the space of tight frames of k vectors lying on the unit sphere of $\mathbf{E}^n$, and let*

$$\mathcal{G}_{k,n}^{\mathbf{E}}=\{\frac{k}{n}P\mid P\in M_k(\mathbf{E})\text{ a projection of rank }n,\ P_{ii}=\frac{n}{k},\ (1\le i\le k)\}.\tag{9}$$

Then the map $\pi=\pi_{k,n}^{\mathbf{E}}:\mathcal{F}_{k,n}^{\mathbf{E}}\to\mathcal{G}_{k,n}^{\mathbf{E}}$ defined by $\pi(F)=F^*F$ is a surjective, locally trivial fiber bundle with fibers $\mathcal{O}_n^{\mathbf{E}}$. Moreover, frames in $\mathcal{F}_{k,n}^{\mathbf{E}}$ have the same image under π if and only if they lie in the same orbit of the action of $\mathcal{O}_n^{\mathbf{E}}$ on $\mathcal{F}_{k,n}^{\mathbf{E}}$. Hence $\mathcal{G}_{k,n}^{\mathbf{E}}$ with the relative topology from $M_k(\mathbf{E})$ is homeomorphic to the space of orbits of the action of $\mathcal{O}_n^{\mathbf{E}}$ on $\mathcal{F}_{k,n}^{\mathbf{E}}$, endowed with the quotient topology.

The following result is now obvious.

Corollary 2.7. *If $k,n\in\mathbf{N}$ with $k>n$, then there is a homeomorphism $\gamma_{k,n}:\mathcal{G}_{k,n}^{\mathbf{E}}\to\mathcal{G}_{k,k-n}^{\mathbf{E}}$ given by $\gamma_{k,n}(\frac{k}{n}P)=\frac{k}{k-n}(I-P)$.*

Remark 2.8. The homeomorphism $\gamma_{k,n}$ intertwines the re–ordering actions of $\mathfrak{S}_k$ on $\mathcal{G}_{k,n}^{\mathbf{E}}$ and $\mathcal{G}_{k,k-n}^{\mathbf{E}}$, described in Remark 2.4. Moreover, $\gamma_{k,n}$ intertwines the diagonal actions of $\mathcal{D}_k^{\mathbf{E}}$ on $\mathcal{G}_{k,n}^{\mathbf{E}}$ and $\mathcal{G}_{k,k-n}^{\mathbf{E}}$, described in Remark 2.5.

3. Frames with one redundant vector

The spherical tight frames of $n+1$ vectors in $\mathbf{R}^n$ are well understood. Goyal, Kovačević and Kelner proved in [7, Thm 2.6] that there is only one of them, up to orthogonal transformations of $\mathbf{R}^n$ and the vector–flipping action of $\mathcal{D}_{n+1}^{\mathbf{R}}$ described in Remark 2.5. The homeomorphism $\gamma_{n+1,1}$ of Corollary 2.7 yields another proof of this theorem, and of the analogous result for $\mathbf{C}^n$.

Theorem 3.1. *Let $n \in \mathbf{N}$. Then*

(i) *$\mathcal{G}^{\mathbf{C}}_{n+1,n}$ is homeomorphic to the n–torus, $\mathbf{T}^n$; morover, the orbit space $\mathcal{G}^{\mathbf{C}}_{n+1,n}/\mathcal{D}^{\mathbf{C}}_{n+1}$ contains only one point;*

(ii) *$\mathcal{G}^{\mathbf{R}}_{n+1,n}$ has exactly 2^n points, $\mathcal{G}^{\mathbf{R}}_{n+1,n}/\mathfrak{S}_{n+1}$ has exactly $[\frac{n}{2}]+1$ points, and $\mathcal{G}^{\mathbf{R}}_{n+1,n}/\mathcal{D}^{\mathbf{R}}_{n+1}$ has only one point.*

Proof. We prove the complex case (i), the real case being similar. The projections of rank 1 in $M_{n+1}(\mathbf{C})$ having all diagonal entries equal to $1/(n+1)$ are in bijective correspondence with the subspaces of $\mathbf{C}^{n+1}$ of the form $\mathbf{C}v$, where

$$v^t = (\tfrac{1}{\sqrt{n+1}}, \tfrac{\zeta_1}{\sqrt{n+1}}, \ldots, \tfrac{\zeta_n}{\sqrt{n+1}}),$$

for $\zeta_1, \ldots, \zeta_n \in \mathbf{T}$. This yields the homeomorphism $\mathbf{T}^n \to \mathcal{G}^{\mathbf{C}}_{n+1,1}$ given by

$$(\zeta_1, \ldots, \zeta_n) \mapsto (\zeta_{i-1}\overline{\zeta_{j-1}})_{1\le i,j,\le n+1} \in M_{n+1}(\mathbf{C}),$$

where we set $\zeta_0 = 1$. Since

$$(\zeta_{i-1}\overline{\zeta_{j-1}})_{1\le i,j,\le n+1} = \mathrm{diag}(1, \zeta_1, \ldots, \zeta_n)\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\mathrm{diag}(1, \overline{\zeta_1}, \ldots, \overline{\zeta_n}),$$

it is clear that the orbit space $\mathcal{G}^{\mathbf{C}}_{n+1,1}/\mathcal{D}^{\mathbf{C}}_{n+1}$ consists of only one point. Now the conclusions in (i) follow from the homeomorphism $\gamma_{n+1,1}$ and Remark 2.8. □

We now write down explicitly a STF, F, of $n+1$ vectors in $\mathbf{R}^n$. Thus, all STFs of $n+1$ vectors in $\mathbf{R}^n$ are obtained from this one by possibly negating some vectors and transforming with an element of $\mathcal{O}^{\mathbf{R}}_n$, and all STFs of $n+1$ vectors in $\mathbf{C}^n$ are obtained from F by multiplying the vectors by unimodular complex numbers and transforming with an element of $\mathcal{O}^{\mathbf{C}}_n$.

Example 3.2. We begin with the frame $F_1 = (1, \ldots, 1) \in \mathcal{F}^{\mathbf{R}}_{n+1,1}$. The corresponding element of $\mathcal{G}^{\mathbf{R}}_{n+1,1}$ is $F_1^* F_1 = (n+1)P$, where P is the projection onto the subspace of $\mathbf{R}^{n+1}$ spanned by $w = (1, \ldots, 1)^t$. Applying $\gamma_{n+1,1}$, we get $\frac{n+1}{n}(I-P) \in \mathcal{G}^{\mathbf{R}}_{n+1,n}$, which corresponds to a frame

$$F = \sqrt{\frac{n+1}{n}}\left(I_n \middle| \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix}\right) V,$$

where $V \in \mathcal{O}^{\mathbf{R}}_{n+1}$ is such that $I - P = V^*\mathrm{diag}(1, \ldots, 1, 0)V$. Thus V has rows $v_1, \ldots, v_{n+1}$, where $v^t_{n+1} = \pm w$ and $v^t_1, \ldots, v^t_n$ can be any orthonormal basis for $w^\perp$.

We choose

$$\begin{aligned}
v_1 &= \tfrac{1}{\sqrt{2}}(1,-1,0,\dots,0)\\
v_2 &= \tfrac{1}{\sqrt{6}}(1,1,-2,0,\dots,0)\\
&\vdots\\
v_j &= \tfrac{1}{\sqrt{j(j+1)}}(\underbrace{1,\dots,1}_{j},-j,0,\dots,0)\\
&\vdots\\
v_n &= \tfrac{1}{\sqrt{n(n+1)}}(1,\dots,1,-n).
\end{aligned}$$

This yields the frame $F=(f_1,\dots,f_{n+1})\in\mathcal{F}_{n+1,n}$, where

$$\begin{aligned}
f_1^t &= \sqrt{\tfrac{n+1}{n}}\Big(\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{12}},\tfrac{1}{\sqrt{20}},\dots,\tfrac{1}{\sqrt{n(n+1)}}\Big)\\
f_2^t &= \sqrt{\tfrac{n+1}{n}}\Big(\tfrac{-1}{\sqrt{2}},\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{12}},\tfrac{1}{\sqrt{20}},\dots,\tfrac{1}{\sqrt{n(n+1)}}\Big)\\
&\vdots\\
f_p^t &= \sqrt{\tfrac{n+1}{n}}\Big(\underbrace{0,\dots,0}_{p-2},\tfrac{-(p-1)}{\sqrt{(p-1)p}},\tfrac{1}{\sqrt{p(p+1)}},\dots,\tfrac{1}{\sqrt{n(n+1)}}\Big),\\
&\vdots\\
f_{n+1}^t &= \sqrt{\tfrac{n+1}{n}}\Big(0,\dots,0,\tfrac{-n}{\sqrt{n(n+1)}}\Big) \qquad\qquad = (0,\dots,0,-1).
\end{aligned}$$

One easliy verifies that all vectors in this frame have the same angle between them:

$$\langle f_p,f_q\rangle = -1/n,\quad (p\neq q).$$

4. Manifold structure

Let $\mathbf{E}=\mathbf{R}$ or $\mathbf{E}=\mathbf{C}$. Let $\odot$ denote the binary operation of coordinate–wise multiplication on $\mathbf{E}^k$. Thus, if $v=(v_i)_{i=1}^k$ and $w=(w_i)_{i=1}^k$, then $v\odot w=(v_iw_i)_{i=1}^k$. Taking $\odot$ as multiplication makes $\mathbf{E}^k$ into the commutative, unital $\mathbf{E}$–algebra that is often denoted $\ell^\infty_{k,\mathbf{E}}$

Let $k,n\in\mathbf{N}$, $k>n$. The Grassman manifold

$$G^{\mathbf{E}}_{k,n}=\{V^*DV\mid V\in\mathcal{O}^{\mathbf{E}}_k\},\qquad D=\mathrm{diag}(\underbrace{1,\dots,1}_{n},0,\dots,0) \tag{10}$$

of n–planes in $\mathbf{E}^k$ is a real analytic submanifold of $M_k(\mathbf{E})$, and we have

$$\frac{n}{k}\mathcal{G}^{\mathbf{E}}_{k,n}=\{P\in G^{\mathbf{E}}_{k,n}\mid P_{ii}=\frac{n}{k},\ (1\le i\le k)\}.$$

Let

$$f : G^{\mathbf{E}}_{k,n} \to K_{k,n} \stackrel{\text{def}}{=} \{(d_i)_{i=1}^k \in \mathbf{R}^k \mid \sum_{i=1}^k d_i = n\}$$

be the map $f(P) = (P_{ii})_{i=1}^k$, that extracts the diagonal of the projection. For $P \in G^{\mathbf{E}}_{k,n}$, we denote as usual the differential of f at P by

$$Df_P : T_P G^{\mathbf{E}}_{k,n} \to T_{f(P)} K_{k,n}. \tag{11}$$

Let $\xi_1, \ldots, \xi_k$ be the standard orthonormal basis for $\mathbf{R}^k$, (also for $\mathbf{C}^k$). Given a subset $A \subseteq \{1, \ldots, k\}$, let $E_A = \sum_{i \in A} \xi_i$ and let $Q_A : \mathbf{E}^k \to \mathbf{E}^k$ be the projection onto the subspace

$$\operatorname{span}\{\xi_i \mid i \in A\}$$

of $\mathbf{E}^k$. Then

$$Q_A(\eta) = E_A \odot \eta, \qquad (\eta \in \mathbf{E}^k). \tag{12}$$

Definition 4.1. Given $T \in M_k(\mathbf{E})$, let σ_T be the set of all minimal nonempty subsets A of $\{1, \ldots, k\}$ that satisfy $Q_A T = T Q_A$. Note that σ_T is a partition of $\{1, \ldots, k\}$.

Lemma 4.2. *Take a projection $P \in G^{\mathbf{E}}_{k,n}$ in the Grassman manifold. Let $\mathcal{W} \subseteq \mathbf{R}^k$ be the range of the differential map* (11)*, and let $\mathcal{W}^\perp$ denote the orthocomplement of $\mathcal{W}$ in $\mathbf{R}^k$. Then a basis for $\mathcal{W}^\perp$ is*

$$\{E_A \mid A \in \sigma_P\}. \tag{13}$$

Consequently, P is a regular point of f if and only if $PQ_A \neq Q_A P$ for all proper, nonempty subsets $A \subseteq \{1, \ldots, k\}$.

Proof. Since P is a regular point of f if and only if $\dim(\mathcal{W}) = k-1$, the last statement of the lemma will follow immediately once (13) is shown to be a basis for $\mathcal{W}^\perp$.

Take $\mathbf{E} = \mathbf{C}$. Let $P = V^* D V$ be as in (10). Let $(e_{\iota j})_{1 \le \iota, j \le k}$ be the standard system of matrix units for $M_k(\mathbf{C})$. A basis for the tangent space $T_P G^{\mathbf{C}}_{k,n}$ is the list of $2n(k-n)$ vectors

$$\big(x(\iota, j)\big)_{1 \le \iota \le n < j \le k}, \quad \big(y(\iota, j)\big)_{1 \le \iota \le n < j \le k},$$

where

$$x(\iota, j) = \frac{d}{dt}\bigg|_{t=0} V^* e^{t(e_{j\iota} - e_{\iota j})} D e^{t(e_{\iota j} - e_{j\iota})} V = V^*(e_{j\iota} + e_{\iota j}) V$$

$$y(\iota, j) = \frac{d}{dt}\bigg|_{t=0} V^* e^{-it(e_{\iota j} + e_{j\iota})} D e^{it(e_{\iota j} + e_{j\iota})} V = iV^*(e_{\iota j} - e_{j\iota}) V.$$

The pth diagonal entries of these are

$$x(\iota, j)_{pp} = 2\,\mathrm{Re}\,(v_{\iota p} \overline{v_{jp}}), \qquad y(\iota, j)_{pp} = 2\,\mathrm{Im}\,(v_{\iota p} \overline{v_{jp}}),$$

where $v_{\iota p}$ is the (ι, p)th entry of V. Let v_ι denote the ιth row of V. Therefore,

$$Df_P(x(\iota, j)) = 2\,\mathrm{Re}\,(v_\iota \odot \overline{v_j}), \qquad Df_P(y(\iota, j)) = 2\,\mathrm{Im}\,(v_\iota \odot \overline{v_j}).$$

Letting $\mathcal{V}=P(\mathbf{C}^k)$, we have

$$\mathcal{V}=\operatorname{span}\{v_\iota^t \mid 1\le\iota\le n\}$$
$$\mathcal{V}^\perp=\operatorname{span}\{v_j^t \mid n<j\le k\}.$$

For $u\in\mathbf{R}^k$, we therefore have

$$\begin{aligned} u\in\mathcal{W}^\perp \quad &\Leftrightarrow \quad \langle u, v\odot\overline{v'}\rangle=0, \quad (v\in\mathcal{V},\, v'\in\mathcal{V}^\perp) \\ &\Leftrightarrow \quad \langle v', u\odot v\rangle=0, \quad (v\in\mathcal{V},\, v'\in\mathcal{V}^\perp) \\ &\Leftrightarrow \quad u\odot\mathcal{V}\subseteq\mathcal{V}. \end{aligned} \tag{14}$$

From (14), we see that $\mathcal{W}^\perp$ is a unital subalgebra of $\ell^\infty_{k,\mathbf{R}}$. It is a standard result, and not difficult to show, that all unital subalgebras of $\ell^\infty_{k,\mathbf{R}}$ are of the form

$$\operatorname{span}_{\mathbf{R}}\{E_A \mid A\in\sigma\}, \tag{15}$$

where σ is a partition of $\{1,\dots,k\}$. But from (14) and (12),

$$E_A\in\mathcal{W}^\perp \quad\Leftrightarrow\quad E_A\odot\mathcal{V}\subseteq\mathcal{V} \quad\Leftrightarrow\quad Q_A(\mathcal{V})\subseteq\mathcal{V} \quad\Leftrightarrow\quad Q_AP=PQ_A.$$

This concludes the proof in the case $\mathbf{E}=\mathbf{C}$.

The proof in the case $\mathbf{E}=\mathbf{R}$ is similar, but easier. Indeed, a basis for the tangent space of $G^{\mathbf{R}}_{k,n}$ is $(x(\iota,j))_{1\le\iota\le n<j\le k}$ and, with $\mathcal{V}=P(\mathbf{R}^k)$, we find that the range of Df_P is

$$\mathcal{W}=\operatorname{span}\{v\odot v' \mid v\in\mathcal{V},\, v'\in\mathcal{V}^\perp\}.$$

Now the proof proceeds as before, beginning with the chain of implications (14). $\square$

Theorem 4.3. *Let $n,k\in\mathbf{N}$, $k>n$, with n and k relatively prime. Then*

(i) *$\mathcal{G}^{\mathbf{R}}_{k,n}$ is a regular, real analytic submanifold of $M_k(\mathbf{R})$ of dimension*

$$\dim(\mathcal{G}^{\mathbf{R}}_{k,n})=(k-n-1)(n-1);$$

(ii) *$\mathcal{F}^{\mathbf{R}}_{k,n}$ is a regular, real analytic submanifold of $(S^{n-1})^k$ of dimension*

$$\dim(\mathcal{F}^{\mathbf{R}}_{k,n})=(k-\frac{n}{2}-1)(n-1);$$

(iii) *$\mathcal{G}^{\mathbf{C}}_{k,n}$ is a regular, real analytic submanifold of $M_k(\mathbf{C})$ of dimension*

$$\dim(\mathcal{G}^{\mathbf{C}}_{k,n})=2n(k-n)-k+1;$$

(iv) *$\mathcal{F}^{\mathbf{C}}_{k,n}$ is a regular, real analytic submanifold of $(S^{2n-1})^k$ of dimension*

$$\dim(\mathcal{F}^{\mathbf{C}}_{k,n})=2n(k-n)+n^2-k+1;$$

Proof. By the proof of Theorem 2.2 and Remark 2.3, (ii) will follow from (i) and (iv) will follow from (iii).

We will show that $c=(\frac{n}{k})_{i=1}^k$ is a regular value of $f:G^{\mathbf{E}}_{k,n}\to K_{k,n}$, for $\mathbf{E}=\mathbf{R}$ and $\mathbf{E}=\mathbf{C}$. Since $\mathcal{G}^{\mathbf{E}}_{k,n}$ is nonempty (see [7], [8] or [6]), by the regular value theorem, this will imply (i) and (iii). By Lemma 4.2, it will suffice to show that if $P\in f^{-1}(c)$, then $PQ_A\ne Q_AP$ for all proper, nonempty subsets A of $\{1,\dots,k\}$. Suppose, to obtain a contradiction, we have $Q_AP=PQ_A$ for some such subset A. Then PQ_A can be viewed as an $|A|\times|A|$ matrix and is a projection, all of whose diagonal entries are

$\frac{n}{k}$. The rank of $Q_A P$ is thus $\frac{n}{k}|A|$. However, since $1 \le |A| \le k-1$ and since n and k are relatively prime, $\frac{n}{k}|A|$ cannot be an integer; this is a contradiction. $\square$

Lemma 4.4. *Let $k, n \in \mathbf{N}$, $k > n$. Then there is $S \in \mathcal{G}^{\mathbf{R}}_{k,n}$ such that $\frac{n}{k}S$ is a regular point of the map f.*

Proof. Let d be the greatest common divisor of k and n. If $d = 1$, then it follows from the proof of Theorem 4.3 that for every $S \in \mathcal{G}^{\mathbf{R}}_{k,n}$, $\frac{n}{k}S$ is a regular point of f. Suppose $d > 1$. Let $k' = k/d$ and $n' = n/d$. Let $R' \in \mathcal{G}^{\mathbf{R}}_{k',n'}$ and let $R = \operatorname{diag}(R', \dots, R')$ be the indicated block diagonal $d \times d$ matrix of $k' \times k'$ matrices. Then $R \in \mathcal{G}^{\mathbf{R}}_{k,n}$. Let $\xi_1, \dots, \xi_k$ be the standard orthonormal basis of $\mathbf{R}^k$ and for $\ell < k$, identify $\mathbf{R}^\ell$ with the usual subspace of $\mathbf{R}^k$, having standard orthonormal basis $\xi_1, \dots, \xi_\ell$. Let $U \in \mathcal{O}^{\mathbf{R}}_d$ be a real orthogonal matrix satisfying

$$\langle U\xi_1, \xi_j \rangle \neq 0, \qquad (1 \le j \le d).$$

Let $W \in \mathcal{O}^{\mathbf{R}}_k$ be any real orthogonal matrix satisfying

$$W\xi_j = \xi_{1+(j-1)k'}, \qquad (1 \le j \le d).$$

Let $V = W\left(\begin{smallmatrix} U & 0 \\ 0 & I_{k-d} \end{smallmatrix}\right)W^* \in \mathcal{O}^{\mathbf{R}}_k$. Then

$$V\xi_p = \begin{cases} \sum_{i=1}^d u_{ij}\xi_{1+(i-1)k'}, & p = 1+(j-1)k',\ 1 \le j \le d, \\ \xi_p, & p \notin \{1,\ 1+k',\ 1+2k',\ \dots,\ 1+(d-1)k'\}, \end{cases}$$

where u_{ij} is the (i,j)th entry of U. Let $S = V^*RV$.

In order to show $S \in \mathcal{G}^{\mathbf{R}}_{k,n}$, it will suffice to show

$$\langle S\xi_p, \xi_p \rangle = 1, \qquad (1 \le p \le k). \tag{16}$$

Suppose $p \notin \{1,\ 1+k',\ 1+2k',\ \dots,\ 1+(d-1)k'\}$. Then

$$\langle S\xi_p, \xi_p \rangle = \langle RV\xi_p, V\xi_p \rangle = \langle R\xi_p, \xi_p \rangle = 1.$$

Suppose $p = 1+(j-1)k'$. Then

$$\begin{aligned} \langle S\xi_p, \xi_p \rangle = \langle RV\xi_p, V\xi_p \rangle &= \sum_{i=1}^d \sum_{i'=1}^d u_{ij}\overline{u_{i'j}}\langle R\xi_{1+(i-1)k'}, \xi_{1+(i'-1)k'} \rangle \\ &= \sum_{i=1}^d |u_{ij}|^2 \langle R\xi_{1+(i-1)k'}, \xi_{1+(i-1)k'} \rangle = \sum_{i=1}^d |u_{ij}|^2 = 1. \end{aligned}$$

Thus (16) is proved.

Consider the relation $\overset{c}{\sim}$ on $\{1, \dots, k\}$ defined by

$$i \overset{c}{\sim} j \quad \Leftrightarrow \quad \langle S\xi_i, \xi_j \rangle \neq 0$$

and let $\sim$ be the equivalence relation on $\{1, \dots, k\}$ generated by $\overset{c}{\sim}$. We will show that $\sim$ has only one equivalence class, which by Lemma 4.2, is equivalent to $\frac{n}{k}S$ being a regular point of f. Let $\overset{c'}{\sim}$ be the relation on $\{1, \dots, k'\}$ defined by

$$i \overset{c'}{\sim} j \quad \Leftrightarrow \quad \langle R'\xi_i, \xi_j \rangle \neq 0.$$

Since k' and n' are relatively prime, from the proof of Theorem 4.3 we have $Q_A R' = R' Q_A$ for all proper, nonempty subsets $A \subseteq \{1, \ldots, k\}$, and therefore, we know that the equivalence relation on $\{1, \ldots, k'\}$ generated by $\overset{c'}{\sim}$ has only one equivalence class. Take $i' \in \{1, \ldots, d\}$ and $s, t \in \{2, \ldots, k'\}$ and set

$$p = s + (i'-1)k', \qquad q = t + (i'-1)k'.$$

Then

$$\langle S\xi_p, \xi_q \rangle = \langle R\xi_p, \xi_q \rangle = \langle R'\xi_s, \xi_t \rangle$$

$$\langle S\xi_1, \xi_q \rangle = \sum_{i=1}^{d} u_{i1} \langle R\xi_{1+(i-1)k'}, \xi_q \rangle = u_{i'1} \langle R\xi_{1+(i'-1)k'}, \xi_q \rangle = u_{i'1} \langle R'\xi_1, \xi_t \rangle.$$

Therefore,

$$s \overset{c'}{\sim} t \quad \Longrightarrow \quad s + (i'-1)k' \overset{c}{\sim} t + (i'-1)k',$$
$$1 \overset{c'}{\sim} t \quad \Longrightarrow \quad 1 \overset{c}{\sim} t + (i'-1)k'.$$

We conclude

$$1 \sim q, \qquad (q \in \{1, \ldots, k\} \backslash \{1,\, 1+k',\, 1+2k',\, \ldots,\, 1+(d-1)k'\}). \tag{17}$$

Finally, there must be $s \in \{2, \ldots, k'\}$ such that $1 \overset{c'}{\sim} s$. Given $i \in \{1, \ldots, d\}$, let $i' \in \{1, \ldots, d\}$ be such that $u_{i'i} \neq 0$ and let

$$p = 1 + (i-1)k', \qquad q = s + (i'-1)k'.$$

Then

$$\langle S\xi_p, \xi_q \rangle = \langle RV\xi_p, \xi_q \rangle = \sum_{j'=1}^{d} u_{j'i} \langle R\xi_{1+(j'-1)k'}, \xi_q \rangle = u_{i'i} \langle R'\xi_1, \xi_s \rangle \neq 0.$$

Thus, we have $p \overset{c}{\sim} q$. But from (17) we have $q \sim 1$, and we conclude $p \sim 1$. Combined with (17), this shows $1 \sim r$ for all $r \in \{1, \ldots, k\}$. $\square$

Definition 4.5. Let $k, n \in \mathbf{N}$ with $k > n$ and let $\mathbf{E} = \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$. Whenever σ is a partition of the set $\{1, \ldots, k\}$, let

$$N^{\mathbf{E}}_{k,n}(\sigma) = \{R \in \mathcal{G}^{\mathbf{E}}_{k,n} \mid \sigma_R = \sigma\},$$

where σ_R is as in Definition 4.1. We will also write simply $N^{\mathbf{E}}_{k,n}$ to denote $N^{\mathbf{E}}_{k,n}(\mathbf{1}_k)$, where $\mathbf{1}_k$ is the trivial partition of $\{1, \ldots, k\}$ into one subset.

Theorem 4.6. *Let $k, n \in \mathbf{N}$ with $k > n$ and let $\mathbf{E} = \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$. Then*

(i) *$N^{\mathbf{E}}_{k,n}$ is a nonempty, regular, real analytic submanifold of $M_k(\mathbf{E})$ with*

$$\dim(N^{\mathbf{E}}_{k,n}) = \begin{cases} (k-n-1)(n-1), & \mathbf{E} = \mathbf{R}, \\ 2n(k-n) - k + 1, & \mathbf{E} = \mathbf{C}. \end{cases}$$

Let $d = \gcd(k, n)$ and let $k' = k/d$, $n' = n/d$. Let $\mathcal{P}(k, k')$ be the set of all partitions of the set $\{1, \ldots, k\}$ into subsets whose cardinalities are multiples of k'. Then

(ii)

$$\mathcal{G}^{\mathbf{E}}_{k,n} = \bigcup_{\sigma \in \mathcal{P}(k,k')} N^{\mathbf{E}}_{k,n}(\sigma) \tag{18}$$

and the sets $(N^{\mathbf{E}}_{k,n}(\sigma))_{\sigma \in \mathcal{P}(k,k')}$ *are pairwise disjoint;*

(iii) *if* $\sigma = \{A_1, \dots, A_\ell\} \in \mathcal{P}(k,k')$ *with* $|A_i| = m_i k'$*, then* $N^{\mathbf{E}}_{k,n}(\sigma)$ *is a nonempty, regular, real analytic submanifold of* $M_k(\mathbf{E})$ *and is real–analytically diffeomorphic to the Cartesian product*

$$\prod_{i=1}^{\ell} N^{\mathbf{E}}_{m_i k', m_i n'}.$$

Proof. By Lemma 4.4, $N^{\mathbf{E}}_{k,n}$ is nonempty. From Lemma 4.2, we have

$$N^{\mathbf{E}}_{k,n} = \{R \in \mathcal{G}^{\mathbf{E}}_{k,n} \mid \frac{n}{k} R \text{ a regular point of } f\}.$$

The regular value theorem now implies (i). Indeed,

$$\mathcal{S} = \{P \in G^{\mathbf{E}}_{k,n} \mid P \text{ is a regular point of } f\}$$

is an open subset and is therefore a regular, real analytic submanifold of the Grassman manifold $G^{\mathbf{E}}_{k,n}$. Now $c = (\frac{n}{k})_{i=1}^{k}$ is a regular value of the restriction of f to $\mathcal{S}$; hence $N^{\mathbf{E}}_{k,n}$ is a regular, real analytic submanifold of $\mathcal{S}$ and thus also of $M_k(\mathbf{E})$.

The assertions of (ii) are clear with the possible exception of the inclusion $\subseteq$ in (18), which we will prove by showing $R \in \mathcal{G}^{\mathbf{E}}_{k,n}$ implies $\sigma_R \in \mathcal{P}(k,k')$. In fact, this is just a variant of the argument used to prove Theorem 4.3. Let $P = \frac{n}{k} R$. If $A \subseteq \{1, \dots, k\}$ and $Q_A P = P Q_A$, then $P Q_A$ is a projection that can be viewed as an $|A| \times |A|$ matrix, all of whose diagonal entries are $\frac{n}{k} = \frac{n'}{k'}$. Hence the rank of PQ_A is $|A| \frac{n'}{k'}$, which must therefore be an integer. Since $\gcd(n', k') = 1$, $|A|$ must be a multiple of k'.

For (iii), we may without loss of generality assume the subsets $A_1, \dots, A_\ell$ are consequetive submintervals of $\{1, \dots, k\}$. Let $R \in \mathcal{G}^{\mathbf{E}}_{k,n}$. Then $R \in N^{\mathbf{E}}_{k,n}$ if and only if R is a block diagonal matrix $R = \operatorname{diag}(R_1, \dots, R_\ell)$, with $R_i \in N^{\mathbf{E}}_{m_i k', m_i n'}$. The assertions of (iii) now follow readily from (i). $\square$

The above result on a manifold stratification structure of $\mathcal{G}^{\mathbf{E}}_{k,n}$, together with the fiber bundle result Theorem 2.2 (and Remark 2.3), yield directly a manifold stratification structure of $\mathcal{F}^{\mathbf{E}}_{k,n}$. However, the following lemma allows us to describe this manifold stratification of $\mathcal{F}^{\mathbf{E}}_{k,n}$ directly in terms of frames.

Lemma 4.7. *Let* $F = (f_1, \dots, f_k) \in \mathcal{F}^{\mathbf{E}}_{k,n}$ *and let* $A \subseteq \{1, \dots, k\}$ *be a subset. Then* F^*F *commutes with* Q_A *if and only if there is a subspace* $\mathcal{V} \subseteq \mathbf{E}^n$ *such that* $(f_i)_{i \in A}$ *forms a spherical tight frame for* $\mathcal{V}$*, while* $(f_i)_{i \in A^c}$ *forms a spherical tight frame for* $\mathcal{V}^\perp$*, where* A^c *is the complement of* A*. Moreover, if* F^*F *commutes with* Q_A*, then the cardinality of* A *is a multiple of* k/d*, where* $d = \gcd(k, n)$*.*

Proof. Multiplying F on the right by a permutation matrix, if necessary, we may without loss of generality assume $A = \{1, \ldots, p\}$ for some $p \in \{1, \ldots, k\}$. Then $F = (F_1|F_2)$, where $F_1 = (f_1, \ldots, f_p)$ and $F_2 = (f_{p+1}, \ldots, f_k)$, and

$$F^*F = \begin{pmatrix} F_1^*F_1 & F_1^*F_2 \\ F_2^*F_1 & F_2^*F_2 \end{pmatrix}.$$

If Q_A commutes with F^*F, then $F_1^*F_2 = 0$. Moreover, since $\sqrt{\frac{n}{k}}F$ is a co–isometry, letting $\mathcal{V}$ be the range of F_1^*, $F_1^*F_2 = 0$ implies that $\sqrt{\frac{n}{k}}F_1^*$ is an isometry from $\mathbf{E}^p$ onto $\mathcal{V}$, while $\sqrt{\frac{n}{k}}F_2^*$ is an isometry from $\mathbf{E}^{k-p}$ onto $\mathcal{V}^\perp$, i.e. F_1 is a spherical tight frame for $\mathcal{V}$ as is F_2 for $\mathcal{V}^\perp$. The converse direction is clear.

An argument showing that $|A|$ must be a multiple of k/d is contained in the proof of part (ii) of Theorem 4.6. □

Definition 4.8. Let $F = (f_i)_{i\in I}$ be a tight frame for some Hilbert space $\mathcal{H}$. We say F is *orthodecomposable* if there is a proper, nonempty subset $A \subseteq I$ such that $(f_i)_{i\in A}$ is a tight frame for some subspace $\mathcal{V}$ of $\mathcal{H}$, and $(f_i)_{i\in A^c}$ is a tight frame for $\mathcal{V}^\perp$.

It is clear from Lemma 4.7 and Definition 4.1 (and, moreover, straighforward to show the analogous result directly in a more general context), that for every $F \in \mathcal{F}^{\mathbf{E}}_{k,n}$, there is a unique partition $\rho_F = \sigma_{F^*F} = \{A_1, \ldots, A_\ell\}$ of the set $\{1, \ldots, k\}$ and there is an orthogonal decomposition $\mathbf{E}^n = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\ell$ such that for every j, $(f_i)_{i\in A_j}$ is a STF for $\mathcal{V}_j$ that is not orthodecomposable.

We set

$$\hat{M}^{\mathbf{E}}_{k,n} = \{F \in \mathcal{F}^{\mathbf{E}}_{k,n} \mid F \text{ not orthodecomposable } \},$$

and, whenever σ is a partition of the set $\{1, \ldots, k\}$, we set

$$\hat{M}^{\mathbf{E}}_{k,n}(\sigma) = \{F \in \mathcal{F}^{\mathbf{E}}_{k,n} \mid \rho_F = \sigma\}.$$

Corollary 4.9. *Let $k, n \in \mathbf{N}$ with $k > n$ and let $\mathbf{E} = \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$. Let $m = n - 1$ if $\mathbf{E} = \mathbf{R}$ and $m = 2n - 1$ if $\mathbf{E} = \mathbf{C}$. Then*

(i) *$\hat{M}^{\mathbf{E}}_{k,n}$ is a nonempty, regular, real analytic submanifold of $(S^m)^k$ with*

$$\dim(\hat{M}^{\mathbf{E}}_{k,n}) = \begin{cases} (k - \frac{n}{2} - 1)(n-1), & \mathbf{E} = \mathbf{R}, \\ 2n(k-n) + n^2 - k + 1, & \mathbf{E} = \mathbf{C}. \end{cases}$$

Let $d = \gcd(k, n)$ and let $k' = k/d$, $n' = n/d$. Let $\mathcal{P}(k, k')$ be the set of all partitions of the set $\{1, \ldots, k\}$ into subsets whose cardinalities are multiples of k'. Then

(ii)

$$\mathcal{F}^{\mathbf{E}}_{k,n} = \bigcup_{\sigma \in \mathcal{P}(k,k')} \hat{M}^{\mathbf{E}}_{k,n}(\sigma)$$

and the sets $(\hat{M}^{\mathbf{E}}_{k,n}(\sigma))_{\sigma\in\mathcal{P}(k,k')}$ are pairwise disjoint;

(iii) *if* $\sigma = \{A_1, \dots, A_\ell\} \in \mathcal{P}(k, k')$ *with* $|A_i| = m_i k'$, *then* $\hat{M}^{\mathbf{E}}_{k,n}(\sigma)$ *is a nonempty, regular, real analytic submanifold of* $(S^m)^k$ *and is real–analytically diffeomorphic to the Cartesian product*

$$\prod_{i=1}^{\ell} \hat{M}^{\mathbf{E}}_{m_i k', m_i n'}.$$

5. The space $\mathcal{G}^{\mathbf{R}}_{4,2}$

In this section we will describe the space $\mathcal{G}^{\mathbf{R}}_{4,2}$ of equivalence classes of spherical tight frames of four vectors in $\mathbf{R}^2$. We begin with some general facts about spherical tight frames of k vectors in $\mathbf{R}^2$.

The following proposition is elementary and well known; *cf* [1, Example 4.2] and [7, Thm 2.7].

Proposition 5.1. *Let* $k \in \mathbf{N}$, $k \geq 2$ *and let* $f_1, \dots, f_k \in \mathbf{R}^2$ *with* $f_1 \neq 0$. *Write* $f_j = \binom{x_j}{y_j}$ *and let* $z_j = x_j + iy_j \in \mathbf{C}$. *Then* $f_1, \dots, f_k$ *is a tight frame for* $\mathbf{R}^2$ *if and only if* $\sum_{j=1}^k z_j^2 = 0$.

Corollary 5.2. *Under the identification of* $\mathbf{R}^2$ *with* $\mathbf{C}$ *used in Proposition 5.1,* $\mathcal{F}^{\mathbf{R}}_{k,2}$ *is identified with*

$$\widetilde{\mathcal{F}}_{k,2} = \{(z_1, \dots, z_k) \in \mathbf{T}^k \mid \sum_{j=1}^k z_j^2 = 0\}.$$

Furthermore, the orbit space $\mathcal{G}^{\mathbf{R}}_{k,2} = \mathcal{F}^{\mathbf{R}}_{k,2}/\mathcal{O}^{\mathbf{R}}_2$ *is identified with the orbit space* $\widetilde{\mathcal{G}}_{k,2}$ *of* $\widetilde{\mathcal{F}}_{k,2}$ *under the group of transformations generated the action of* $\mathbf{Z}_2$ *by complex conjugation,*

$$(z_1, \dots, z_k) \mapsto (\overline{z_1}, \dots, \overline{z_k})$$

and the action of $\mathbf{T}$ *by rotations,*

$$e^{i\theta} \cdot (z_1, \dots, z_k) = (e^{i\theta} z_1, \dots, e^{i\theta} z_k).$$

Since any four elements of $\mathbf{T}$ that sum to zero can be divided into pairs that are negatives of each other, from Corollary 5.2 we easily prove the well–known result that any spherical tight frame of four vectors in $\mathbf{R}^2$ consists of two orthonormal bases. Thus, the orbit space $\widetilde{\mathcal{F}}_{4,2}/\mathbf{T}$ is the union of the images of the twelve maps

$$\left(\sigma_{p,\epsilon_1,\epsilon_2}\right)_{2 \leq p \leq 4,\, \epsilon_1,\epsilon_2 \in \{\pm 1\}},$$

where $\sigma_{p,\epsilon_1,\epsilon_2} : \mathbf{T} \to \widetilde{\mathcal{F}}_{4,2}/\mathbf{T}$ are given by

$$\begin{aligned}
\sigma_{2,\epsilon_1,\epsilon_2}(\zeta) &= [(1, \epsilon_1 i, \zeta, \epsilon_2 i\zeta)] \\
\sigma_{3,\epsilon_1,\epsilon_2}(\zeta) &= [(1, \zeta, \epsilon_1 i, \epsilon_2 i\zeta)] \\
\sigma_{4,\epsilon_1,\epsilon_2}(\zeta) &= [(1, \zeta, \epsilon_2 i\zeta, \epsilon_1 i)].
\end{aligned}$$

For every p we have

$$\overline{\sigma_{p,\epsilon_1,\epsilon_2}(\zeta)} = \sigma_{p,-\epsilon_1,-\epsilon_2}(\overline{\zeta}).$$

Thus $\widetilde{\mathcal{G}}_{4,2} = (\widetilde{\mathcal{F}}_{4,2}/\mathbf{T})/\mathbf{Z}_2$ is the union of the images of the six maps

$$\big(\tau_{p,\epsilon}\big)_{2\le p\le 4,\,\epsilon\in\{\pm 1\}}, \tag{19}$$

where $\tau_{p,\epsilon} : \mathbf{T} \to \widetilde{\mathcal{G}}_{4,2}$ and $\tau_{p,\epsilon}(\zeta) = [\mu_{p,\epsilon}(\zeta)]$ with

$$\begin{aligned}
\mu_{2,\epsilon}(\zeta) &= (1, i, \zeta, \epsilon i\zeta) \\
\mu_{3,\epsilon}(\zeta) &= (1, \zeta, i, \epsilon i\zeta) \\
\mu_{4,\epsilon}(\zeta) &= (1, \zeta, \epsilon i\zeta, i).
\end{aligned}$$

Note we have $\tau_{p,\epsilon}(\zeta) = \tau_{p',\epsilon'}(\zeta')$ if and only if either $\mu_{p,\epsilon}(\zeta) = \mu_{p',\epsilon'}(\zeta')$ or $\mu_{p,\epsilon}(\zeta) = \overline{\mu_{p',\epsilon'}(\zeta')}$. For brevity, we will abreviate $\epsilon = \pm 1$ by $\epsilon = \pm$. For each p and ϵ, we have

$$\begin{aligned}
\mu_{p,\epsilon}(\zeta) = \mu_{p,\epsilon}(\zeta') \qquad &\Leftrightarrow \quad \zeta = \zeta' \\
\mu_{p,\epsilon}(\zeta) = \overline{\mu_{p,\epsilon}(\zeta')} \quad &\text{never happens.}
\end{aligned}$$

Thus each $\tau_{p,\epsilon}$ is injective. It follows that $\widetilde{\mathcal{G}}_{4,2}$ is the indentification space of six circles, glued together according to how the images of the six maps (19) overlap. For each p,

$$\begin{aligned}
\mu_{p,+}(\zeta) = \mu_{p,-}(\zeta') \quad &\text{never happens} \\
\mu_{p,+}(\zeta) = \overline{\mu_{p,-}(\zeta')} \quad &\text{never happens}
\end{aligned}$$

so $\tau_{p,+}$ and $\tau_{p,-}$ have disjoint images. Furthermore,

$$\begin{aligned}
\mu_{2,\epsilon}(\zeta) = \mu_{3,\epsilon'}(\zeta') \quad &\Leftrightarrow \quad \zeta = \zeta' = i,\ \epsilon = \epsilon' \\
\mu_{2,\epsilon}(\zeta) = \overline{\mu_{3,\epsilon'}(\zeta')} \quad &\Leftrightarrow \quad \zeta = \zeta' = -i,\ \epsilon = \epsilon' \\
\mu_{2,\epsilon}(\zeta) = \mu_{4,\epsilon'}(\zeta') \quad &\Leftrightarrow \quad \zeta = -\epsilon',\ \zeta' = i,\ \epsilon = -\epsilon' \\
\mu_{2,\epsilon}(\zeta) = \overline{\mu_{4,\epsilon'}(\zeta')} \quad &\Leftrightarrow \quad \zeta = \epsilon',\ \zeta' = -i,\ \epsilon = -\epsilon' \\
\mu_{3,\epsilon}(\zeta) = \mu_{4,\epsilon'}(\zeta') \quad &\Leftrightarrow \quad \zeta = \zeta' = \epsilon = \epsilon' \\
\mu_{3,\epsilon}(\zeta) = \overline{\mu_{4,\epsilon'}(\zeta')} \quad &\Leftrightarrow \quad \zeta = \zeta' = -\epsilon,\ \epsilon = \epsilon'.
\end{aligned}$$

Representing the image of $\tau_{p,\epsilon}$ as a graph in the obvious way depicted in Figure 1, the only identifications that occur in $\widetilde{\mathcal{G}}_{4,2}$ are among the labeled vertices. In particular, we find that $\widetilde{\mathcal{G}}_{4,2}$ is homeomorphic to the quotient graph whose vertices are

$$\begin{array}{lll}
v_1 = \{\tau_{2,+}(1), \tau_{4,-}(i)\} & v_2 = \{\tau_{2,-}(1), \tau_{4,+}(-i)\} & v_3 = \{\tau_{2,+}(i), \tau_{3,+}(i)\} \\
v_4 = \{\tau_{2,-}(i), \tau_{3,-}(i)\} & v_5 = \{\tau_{2,+}(-1), \tau_{4,-}(-i)\} & v_6 = \{\tau_{2,-}(-1), \tau_{4,+}(i)\} \\
v_7 = \{\tau_{2,+}(-i), \tau_{3,+}(-i)\} & v_8 = \{\tau_{2,-}(-i), \tau_{3,-}(-i)\} & v_9 = \{\tau_{3,+}(1), \tau_{4,+}(1)\} \\
v_{10} = \{\tau_{3,-}(1), \tau_{4,-}(1)\} & v_{11} = \{\tau_{3,+}(-1), \tau_{4,+}(-1)\} & v_{12} = \{\tau_{3,-}(-1), \tau_{4,-}(-1)\},
\end{array}$$

and where each vertex has four edges, surviving from the original graphs in Figure 1. In conclusion:

Theorem 5.3. *the space $\mathcal{G}^{\mathbf{R}}_{4,2}$ is homeomorphic to the graph with twelve vertices and twenty–four edges that is depicted in Figure 2.*

6. The space $\mathcal{G}^{\mathbf{R}}_{5,2}$

In this section we will describe the space $\mathcal{G}^{\mathbf{R}}_{5,2}$ of equivalence classes of tight spherical frames of five vectors in $\mathbf{R}^2$. By Corollary 5.2, $\mathcal{F}^{\mathbf{R}}_{5,2}$ is homeomorphic to $\widetilde{\mathcal{F}}_{5,2}$. We have

$$\mathcal{E} \stackrel{\text{def}}{=} \widetilde{\mathcal{F}}_{5,2}/\mathbf{T} = \{(z_1, z_2, z_3, z_4) \in \mathbf{T}^4 \mid \sum_{j=1}^{4} z_j^2 = -1\}$$

and $\mathcal{G}^{\mathbf{R}}_{5,2}$ is homeomorphic to $\mathcal{E}/\mathbf{Z}_2$ where $\mathbf{Z}_2$ acts on $\mathcal{E}$ by complex conjugation. Let

$$\mathcal{D} = \{(w_1, w_2, w_3, w_4) \in \mathbf{T}^4 \mid \sum_{j=1}^{4} w_j = -1\}$$

and let $p : \mathcal{E} \to \mathcal{D}$ be $p(z_1, z_2, z_3, z_4) = (z_1^2, z_2^2, z_3^2, z_4^2)$. Then p is a sixteen–fold covering map and p intertwines complex conjugation with complex conjugation.

Let

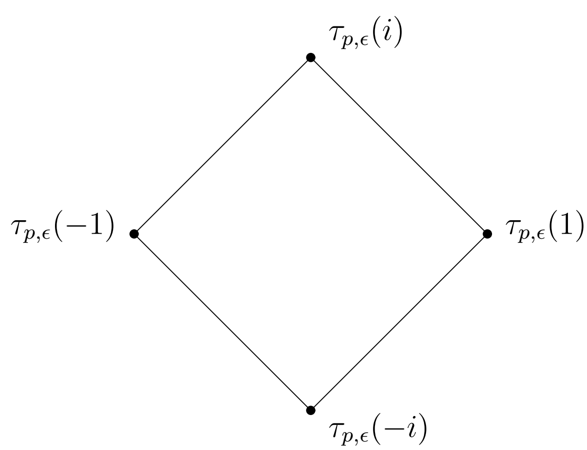
$$A = \{a \in \mathbf{C} \mid 0 < |a| \le 2,\ 0 < |-1-a| \le 2\}.$$

The set A is pictured in Figure 3, where of course we have $-1, 0 \notin A$. Let B be the topological space obtained from A by dilating the punctures at -1 and 0 and gluing copies C_{-1}, respectively C_0, of the circle onto the boundaries of the resulting holes. The space B is pictured in Figure 4. More formally, as a set we let B be the disjoint union of the ranges of three injective maps,

$$\alpha : A \to B, \qquad \tau_{-1} : \mathbf{T} \to B, \qquad \tau_0 : \mathbf{T} \to B,$$

where C_j is the image of τ_j, and where the topology of B is defined as follows:

Figure 1. The image of $\tau_{p,\epsilon}$ as a graph.



- a neighborhood of $\alpha(a)$ in B is a subset of B containing
$$\alpha(\{a' \in A \mid |a'-a| < \epsilon\})$$
for some $\epsilon > 0$;
- a neighborhood of $\tau_0(e^{i\theta})$ in B is a subset of B containing
$$\{\tau_0(e^{i\theta'}) \mid \theta' \in \mathbf{R},\, |\theta'-\theta| < \epsilon\}\cup \\ \cup\,\{\alpha(re^{i\theta'}) \mid \theta' \in \mathbf{R},\, |\theta'-\theta| < \epsilon,\, 0 < r < \epsilon\}$$
for some $0 < \epsilon < 1$;
- a neighborhood of $\tau_{-1}(e^{i\theta})$ in B is a subset of B containing
$$\{\tau_{-1}(e^{i\theta'}) \mid \theta' \in \mathbf{R},\, |\theta'-\theta| < \epsilon\}\cup \\ \cup\,\{\alpha(-1+re^{i\theta'}) \mid \theta' \in \mathbf{R},\, |\theta'-\theta| < \epsilon,\, 0 < r < \epsilon\}$$
for some $0 < \epsilon < 1$.

In Figure 4, we have $q = \alpha(\frac{1}{2} + \frac{\sqrt{15}}{2}i)$ and $\overline{q} = \alpha(\frac{1}{2} - \frac{\sqrt{15}}{2}i)$. Then B is a compact Hausdorff space. Let $p_A : B \to \mathrm{cl}(A)$, where $\mathrm{cl}(A) = A \cup \{0, 1\}$ denotes the closure

FIGURE 2. The space $\mathcal{G}^{\mathbf{R}}_{4,2}$.

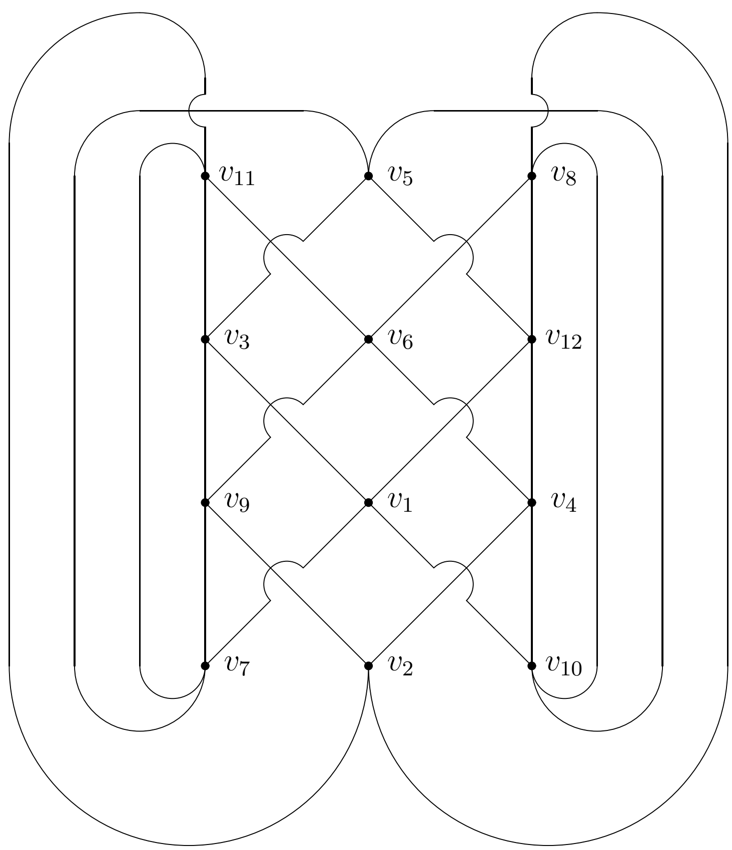


FIGURE 3. The set A.

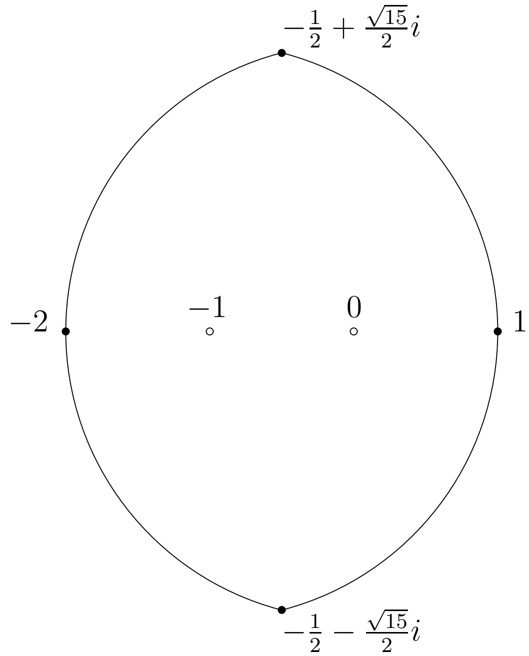
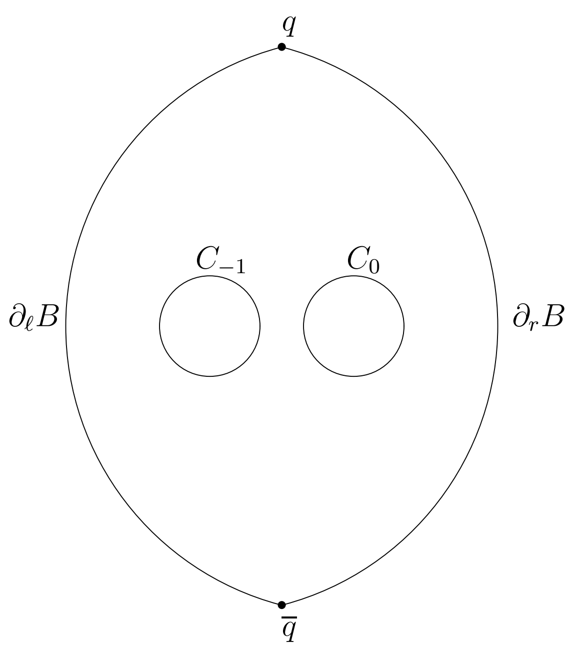


FIGURE 4. The space B.



of A, be the continuous map given by

$$p_A(\alpha(a)) = a, \qquad p_A(\tau_{-1}(\zeta)) = -1, \qquad p_A(\tau_0(\zeta)) = 0, \qquad (\zeta \in \mathbf{T}).$$

Given $a \in A$, there are precisely two values of $w \in \mathbf{T}$ such that there exists $w' \in \mathbf{T}$ with $w + w' = a$. These are illustrated in Figure 5. Let $w_\ell(a)$ be the one of these two values that lies to the left as one travels the line segment from 0 to a, and let $w_r(a)$ be the other value, lying to the right. In Figure 5, $w_\ell(a)$ is the point labeled w' and $w_r(a)$ is the point labeled w. Note we have

$$w_\ell(\overline{a}) = \overline{w_r(a)}, \qquad w_r(\overline{a}) = \overline{w_\ell(a)}. \tag{20}$$

Let $\phi : B \to \mathcal{D}$ be given by

$$\phi(\alpha(a)) = (w_\ell(a), w_r(a), w_\ell(-1-a), w_r(-1-a)), \qquad (a \in A) \tag{21}$$

$$\phi(\tau_0(\zeta)) = (i\zeta, -i\zeta, -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i, -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i) \qquad (\zeta \in \mathbf{T}). \tag{22}$$

$$\phi(\tau_{-1}(\zeta)) = (-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i, -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, -i\zeta, i\zeta) \qquad (\zeta \in \mathbf{T}) \tag{23}$$

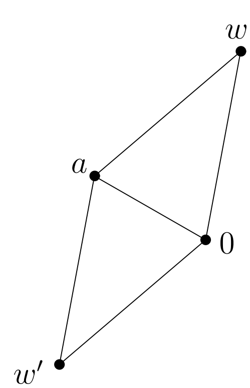
Then ϕ is injective and continuous, hence a homeomorphism onto its image. Let $V = \{e, t, u, v\}$ be the Klein 4–group with e the identity element and let V act on $\mathbf{T}^4$ by

$$\begin{aligned} t \odot (\zeta_1, \zeta_2, \zeta_3, \zeta_4) &= (\zeta_2, \zeta_1, \zeta_3, \zeta_4) \\ u \odot (\zeta_1, \zeta_2, \zeta_3, \zeta_4) &= (\zeta_1, \zeta_2, \zeta_4, \zeta_3) \\ v \odot (\zeta_1, \zeta_2, \zeta_3, \zeta_4) &= (\zeta_2, \zeta_1, \zeta_4, \zeta_3). \end{aligned}$$

For $g \in V$, let $\phi_g : B \to \mathcal{D}$ be $\phi_g(b) = g \odot (\phi(b))$. Then

$$\mathcal{D} = \bigcup_{g \in V} \phi_g(B). \tag{24}$$

FIGURE 5. $w + w' = a$.



Let

$$\partial_\ell B = \{\alpha(a) \mid a \in A,\, |a| = 2\}$$
$$\partial_r B = \{\alpha(a) \mid a \in A,\, |-1-a| = 2\}$$

be the left and right boundaries of B, as indicated in Figure 4. Thus $\partial_\ell B \cap \partial_r B = \{q, \overline{q}\}$.

Lemma 6.1. *Let $b, b' \in B$, $g, g' \in V$. Then*

$$\phi_g(b) = \phi_{g'}(b') \tag{25}$$

if and only if at least one of the following holds:

(i) $b = b'$ *and* $g = g'$
(ii) $b = b' \in \partial_\ell B \cap \partial_r B$
(iii) $b = b' \in \partial_\ell B$ *and* $g^{-1}g' = t$
(iv) $b = b' \in \partial_r B$ *and* $g^{-1}g' = u$
(v) $b = \tau_0(\zeta)$ *and* $b' = \tau_0(-\zeta)$ *for some* $\zeta \in \mathbf{T}$, *and* $g^{-1}g' = t$
(vi) $b = \tau_{-1}(\zeta)$ *and* $b' = \tau_{-1}(-\zeta)$ *for some* $\zeta \in \mathbf{T}$, *and* $g^{-1}g' = u$.

Proof. We write $\phi(b) = (w_1(b), w_2(b), w_3(b), w_4(b))$ for $b \in B$. Then $w_1(b) + w_2(b) = p_A(b)$, so (25) implies $p_A(b) = p_A(b')$, and one of the following holds:

$$b = b' = \alpha(a), \quad \text{some } a \in A \tag{26}$$
$$b, b' \in C_0 \tag{27}$$
$$b, b' \in C_{-1}. \tag{28}$$

Suppose (26) holds. Since $w_1(b) = w_2(b)$ if and only if $b \in \partial_\ell(B)$ and $w_3(b) = w_4(b)$ if and only if $b \in \partial_r(B)$, we quickly deduce that at least one of conditions (i)–(iv) holds, and conversely, any of (i)–(iv) implies (25). If (27) holds, then using (22) we get (25)$\Leftrightarrow$(i) or (v), and if (28) holds, then using (23) we get (25)$\Leftrightarrow$(i) or (vi). $\square$

By (24) and the fact that each ϕ_g is injective, the space $\mathcal{D}$ is obtained by gluing together four copies of B according to how their images under ϕ_e, ϕ_t, ϕ_u and ϕ_v overlap, as described in Lemma 6.1. We will not pursue this, but we will use a similar reasoning to investigate the space $\mathcal{E}/\mathbf{Z}_2$.

Let $\beta : B \to B$ be the homeomorphism of order two defined by complex conjugation, namely

$$\beta(\alpha(a)) = \alpha(\overline{a}) \qquad (a \in A)$$
$$\beta(\tau_{-1}(\zeta)) = \tau_{-1}(\overline{\zeta}) \qquad (\zeta \in \mathbf{T})$$
$$\beta(\tau_0(\zeta)) = \tau_0(\overline{\zeta}) \qquad (\zeta \in \mathbf{T}).$$

Then $\beta^2 = \mathrm{id}$. If $b \in B$ and $\phi(b) = (w_1, w_2, w_3, w_4)$, then from (20)–(22) we see

$$\phi(\beta(b)) = (\overline{w_2}, \overline{w_1}, \overline{w_4}, \overline{w_3}) = \overline{\phi_v(b)}. \tag{29}$$

Let $\widetilde{B}$ be the upper half of B, namely

$$\begin{aligned}\widetilde{B} = \{\alpha(a) \mid a \in A,\ \mathrm{Re}\, a \geq 0\} \cup \\ \cup \{\tau_{-1}(\zeta) \mid \zeta \in \mathbf{T},\ \mathrm{Re}\, \zeta \geq 0\} \cup \\ \cup \{\tau_0(\zeta) \mid \zeta \in \mathbf{T},\ \mathrm{Re}\, \zeta \geq 0\}.\end{aligned}$$

Then $B = \widetilde{B} \cup \beta(\widetilde{B})$.

For future use, we want to consider also the homeomorphism $\gamma : B \to B$ given by

$$\begin{aligned}\gamma(\alpha(a)) &= \alpha(-1-\overline{a}) \quad (a \in A)\\ \gamma(\tau_0(\zeta)) &= \tau_{-1}(-\overline{\zeta}) \quad (\zeta \in \mathbf{T})\\ \gamma(\tau_{-1}(\zeta)) &= \tau_0(-\overline{\zeta}) \quad (\zeta \in \mathbf{T}).\end{aligned}$$

Then $\gamma^2 = \mathrm{id}$. Moreover, for $b \in B$, if $\phi(b) = (w_1, w_2, w_3, w_4)$, then

$$\phi(\gamma(b)) = (\overline{w_4}, \overline{w_3}, \overline{w_2}, \overline{w_1}). \tag{30}$$

We have

$$\phi(q) = \left(-\tfrac{1}{4} + \tfrac{\sqrt{15}}{4}i, -\tfrac{1}{4} + \tfrac{\sqrt{15}}{4}i, -\tfrac{1}{4} - \tfrac{\sqrt{15}}{4}i, -\tfrac{1}{4} - \tfrac{\sqrt{15}}{4}i\right).$$

By homotopy lifting, there is a unique continuous map $\tilde{\phi} : \widetilde{B} \to \mathcal{E}$ such that $p \circ \tilde{\phi} = \phi\restriction_{\widetilde{B}}$ and such that

$$\tilde{\phi}(q) = \left(\sqrt{\tfrac{3}{8}} + \sqrt{\tfrac{5}{8}}\,i, \sqrt{\tfrac{3}{8}} + \sqrt{\tfrac{5}{8}}\,i, \sqrt{\tfrac{3}{8}} - \sqrt{\tfrac{5}{8}}\,i, \sqrt{\tfrac{3}{8}} - \sqrt{\tfrac{5}{8}}\,i\right).$$

Let E be the multiplicative subgroup

$$E = \{(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \mid \epsilon_j \in \{\pm 1\}\}$$

of $\mathbf{T}^4$. Given $g \in V$ and $\epsilon \in E$, let $\tilde{\phi}_g^\epsilon : \widetilde{B} \to \mathcal{E}$ be

$$\tilde{\phi}_g^\epsilon(b) = \epsilon \cdot (g \odot (\tilde{\phi}(b))).$$

Then for any $g \in V$, $(\tilde{\phi}_g^\epsilon)_{\epsilon \in E}$ are the sixteen different liftings of $\phi_g\restriction_{\widetilde{B}}$ to $\mathcal{E}$ (under the covering projection p). in particular, for $b \in \widetilde{B}$,

$$p^{-1}(\phi_g(b)) = \{\tilde{\phi}_g^\epsilon(b) \mid \epsilon \in E\}. \tag{31}$$

If $b \in \beta(\widetilde{B})$, then by (29),

$$p(\overline{\tilde{\phi}_g^\epsilon(\beta(b))}) = \overline{p(\tilde{\phi}_g^\epsilon(\beta(b)))} = \overline{\phi_g(\beta(b))} = \phi_{vg}(b).$$

Therefore,

$$p^{-1}(\phi_{vg}(b)) = \{\overline{\tilde{\phi}_g^\epsilon(\beta(b))} \mid \epsilon \in E\}. \tag{32}$$

Let $\psi_g^\epsilon : \widetilde{B} \to \mathcal{E}/\mathbf{Z}_2$ be $\tilde{\phi}_g^\epsilon$ followed by the quotient map $\mathcal{E} \to \mathcal{E}/\mathbf{Z}_2$ of $\mathcal{E}$ under the action of complex conjugation. Because $B = \widetilde{B} \cup \beta(\widetilde{B})$, from (31) and (32), we get that $\mathcal{E}/\mathbf{Z}_2$ is covered by the images of the sixty–four maps $(\psi_g^\epsilon)_{g \in V, \epsilon \in E}$. Our goal is to understand how these images overlap.

We shall compute the values of $\tilde{\phi}$ on the boundary of $\widetilde{B}$. The space $\widetilde{B}$ is pictured and its boundary labeled in Figure 6. In particular we have the indicated points

$$q = \alpha(-\tfrac{1}{2} + \tfrac{\sqrt{15}}{2} i)$$
$$s = \alpha(-2), \quad r = \alpha(1)$$
$$x_j = \tau_j(-1), \quad y_j = \tau_j(1), \quad (j \in \{-1, 0\})$$

and the (closed) intervals and arcs

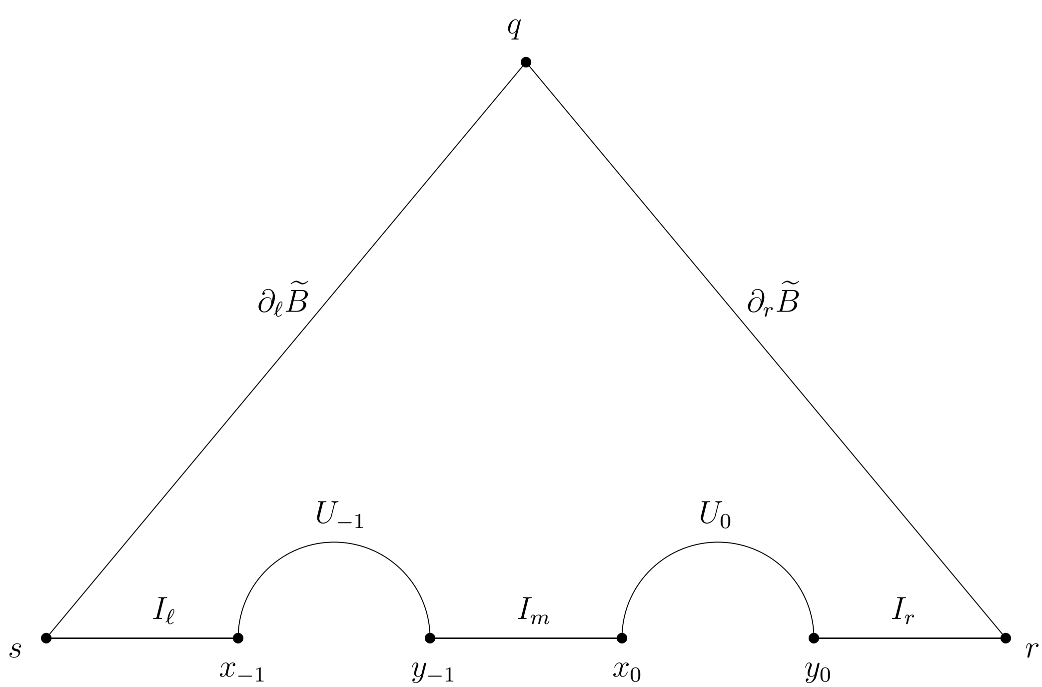
$$\begin{aligned}
\partial_z \widetilde{B} &= \partial_z B \cap \widetilde{B}, \quad (z = \ell, r) \\
U_j &= C_j \cap \widetilde{B}, \quad (j \in \{-1, 0\}) \\
I_\ell &= \alpha([-2, -1)) \cup \{x_{-1}\} \\
I_m &= \{y_{-1}\} \cup \alpha((-1, 0)) \cup \{x_0\} \\
I_r &= \{y_0\} \cup \alpha((0, 1]).
\end{aligned}$$

Let us first determine $\tilde{\phi}(\alpha(-\frac{1}{2} + it))$ for $t \in [0, \frac{\sqrt{15}}{2}]$. Let $b = \alpha(-\frac{1}{2} + it)$ and write

$$\phi(b) = (w_1(b), w_2(b), w_3(b), w_4(b)), \qquad \tilde{\phi}(b) = (z_1(b), z_2(b), z_3(b), z_4(b)).$$

As t descends from $\frac{\sqrt{15}}{2}$ to 0, $w_1(b)$ moves from $-\frac{1}{4} + \frac{\sqrt{15}}{4} i$ to $-\frac{1}{4} - \frac{\sqrt{15}}{4} i$, avoiding the first quadrant, and $w_2(b)$ moves from $-\frac{1}{4} + \frac{\sqrt{15}}{4} i$ back to $-\frac{1}{4} + \frac{\sqrt{15}}{4} i$, staying in the upper half–plane. Therefore, $z_1(b)$ changes from $\sqrt{\frac{3}{8}} + \sqrt{\frac{5}{8}}\, i$ to $-\sqrt{\frac{3}{8}} + \sqrt{\frac{5}{8}}\, i$ and $z_2(b)$ moves from $\sqrt{\frac{3}{8}} + \sqrt{\frac{5}{8}}\, i$ back to $\sqrt{\frac{3}{8}} + \sqrt{\frac{5}{8}}\, i$. Finally, since $b = \gamma(b)$, from (30) we have $w_3(b) = \overline{w_2(b)}$ and $w_4(b) = \overline{w_1(b)}$; consequently $z_3(b) = \overline{z_2(b)}$ and $z_4(b) = \overline{z_1(b)}$.

FIGURE 6. The space $\widetilde{B}$ with labeled boundary.



Hence

$$\tilde\phi(\alpha(-\tfrac12))=\Big(-\sqrt{\tfrac38}+\sqrt{\tfrac58}\,i,\ \sqrt{\tfrac38}+\sqrt{\tfrac58}\,i,\ \sqrt{\tfrac38}-\sqrt{\tfrac58}\,i,\ -\sqrt{\tfrac38}-\sqrt{\tfrac58}\,i\Big).$$

Now it is easy to calculate the values of ϕ and $\tilde\phi$ on the bottom part of the boundary of $\widetilde{B}$. On I_m, for $t\in(-1,0)$,

$$\begin{aligned}\phi(\alpha(t))&=\Big(\tfrac t2-\tfrac{\sqrt{4-t^2}}{2}\,i,\ \tfrac t2+\tfrac{\sqrt{4-t^2}}{2}\,i,\ \tfrac{-1-t}{2}-\tfrac{\sqrt{4-(1+t)^2}}{2}\,i,\ \tfrac{-1-t}{2}+\tfrac{\sqrt{4-(1+t)^2}}{2}\,i\Big)\\ \tilde\phi(\alpha(t))&=\Big(-\sqrt{\tfrac12+\tfrac t4}+\sqrt{\tfrac12-\tfrac t4}\,i,\ \sqrt{\tfrac12+\tfrac t4}+\sqrt{\tfrac12-\tfrac t4}\,i,\\ &\qquad\sqrt{\tfrac12-\tfrac{t+1}{4}}-\sqrt{\tfrac12+\tfrac{t+1}{4}}\,i,\ -\sqrt{\tfrac12-\tfrac{t+1}{4}}-\sqrt{\tfrac12+\tfrac{t+1}{4}}\,i\Big).\end{aligned}\tag{33}$$

Hence, we have

$$\begin{aligned}\phi(y_{-1})&=\Big(-\tfrac12-\tfrac{\sqrt3}{2}i,\ -\tfrac12+\tfrac{\sqrt3}{2}i,\ -i,\ i\Big)\\ \tilde\phi(y_{-1})&=\Big(-\tfrac12+\tfrac{\sqrt3}{2}i,\ \tfrac12+\tfrac{\sqrt3}{2}i,\ \tfrac{1}{\sqrt2}-\tfrac{1}{\sqrt2}i,\ -\tfrac{1}{\sqrt2}-\tfrac{1}{\sqrt2}i\Big)\end{aligned}\tag{34}$$

$$\begin{aligned}\phi(x_0)&=\Big(-i,\ i,\ -\tfrac12-\tfrac{\sqrt3}{2}i,\ -\tfrac12+\tfrac{\sqrt3}{2}i\Big)\\ \tilde\phi(x_0)&=\Big(-\tfrac{1}{\sqrt2}+\tfrac{1}{\sqrt2}i,\ \tfrac{1}{\sqrt2}+\tfrac{1}{\sqrt2}i,\ \tfrac12-\tfrac{\sqrt3}{2}i,\ -\tfrac12-\tfrac{\sqrt3}{2}i\Big).\end{aligned}\tag{35}$$

On U_0, for $0\le\theta\le\pi$,

$$\begin{aligned}\phi(\tau_0(e^{i\theta}))&=\Big(e^{i(\theta+\frac\pi2)},\ e^{i(\theta-\frac\pi2)},\ -\tfrac12-\tfrac{\sqrt3}{2}i,\ -\tfrac12+\tfrac{\sqrt3}{2}i\Big)\\ \tilde\phi(\tau_0(e^{i\theta}))&=\Big(e^{i(\frac\theta2+\frac\pi4)},\ e^{i(\frac\theta2-\frac\pi4)},\ \tfrac12-\tfrac{\sqrt3}{2}i,\ -\tfrac12-\tfrac{\sqrt3}{2}i\Big).\end{aligned}\tag{36}$$

In particular,

$$\begin{aligned}\phi(y_0)&=\Big(i,\ -i,\ -\tfrac12-\tfrac{\sqrt3}{2}i,\ -\tfrac12+\tfrac{\sqrt3}{2}i\Big)\\ \tilde\phi(y_0)&=\Big(\tfrac{1}{\sqrt2}+\tfrac{1}{\sqrt2}i,\ \tfrac{1}{\sqrt2}-\tfrac{1}{\sqrt2}i,\ \tfrac12-\tfrac{\sqrt3}{2}i,\ -\tfrac12-\tfrac{\sqrt3}{2}i\Big).\end{aligned}\tag{37}$$

On I_r, for $t\in(0,1]$,

$$\begin{aligned}\phi(\alpha(t))&=\Big(\tfrac t2+\tfrac{\sqrt{4-t^2}}{2}\,i,\ \tfrac t2-\tfrac{\sqrt{4-t^2}}{2}\,i,\ \tfrac{-1-t}{2}-\tfrac{\sqrt{4-(1+t)^2}}{2}\,i,\ \tfrac{-1-t}{2}+\tfrac{\sqrt{4-(1+t)^2}}{2}\,i\Big)\\ \tilde\phi(\alpha(t))&=\Big(\sqrt{\tfrac12+\tfrac t4}+\sqrt{\tfrac12-\tfrac t4}\,i,\ \sqrt{\tfrac12+\tfrac t4}-\sqrt{\tfrac12-\tfrac t4}\,i,\\ &\qquad\sqrt{\tfrac12-\tfrac{t+1}{4}}-\sqrt{\tfrac12+\tfrac{t+1}{4}}\,i,\ -\sqrt{\tfrac12-\tfrac{t+1}{4}}-\sqrt{\tfrac12+\tfrac{t+1}{4}}\,i\Big).\end{aligned}\tag{38}$$

In particular,

$$\begin{aligned}\phi(r)&=\Big(\tfrac12+\tfrac{\sqrt3}{2}i,\ \tfrac12-\tfrac{\sqrt3}{2}i,-1,-1\Big)\\ \tilde\phi(r)&=\Big(\tfrac{\sqrt3}{2}+\tfrac12 i,\ \tfrac{\sqrt3}{2}-\tfrac12 i,\ -i,\ -i\Big).\end{aligned}\tag{39}$$

On U_{-1}, for $0 \le \theta \le \pi$,

$$\phi(\tau_{-1}(e^{i\theta})) = \Big(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i,\ -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\ e^{i(\theta-\frac{\pi}{2})},\ e^{i(\theta+\frac{\pi}{2})}\Big).$$
$$\tilde{\phi}(\tau_{-1}(e^{i\theta})) = \Big(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\ \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\ e^{i(\frac{\theta}{2}-\frac{\pi}{4})},\ e^{i(\frac{\theta}{2}-\frac{3\pi}{4})}\Big). \tag{40}$$

In particular,

$$\phi(x_{-1}) = \Big(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i,\ -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\ i,\ -i\Big)$$
$$\tilde{\phi}(x_{-1}) = \Big(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\ \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\ \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i,\ \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\Big). \tag{41}$$

On I_ℓ, for $t \in [-2,-1)$,

$$\phi(\alpha(t)) = \Big(\tfrac{t}{2} - \tfrac{\sqrt{4-t^2}}{2}\, i,\ \tfrac{t}{2} + \tfrac{\sqrt{4-t^2}}{2}\, i,\ \tfrac{-1-t}{2} + \tfrac{\sqrt{4-(1+t)^2}}{2}\, i,\ \tfrac{-1-t}{2} - \tfrac{\sqrt{4-(1+t)^2}}{2}\, i\Big)$$
$$\begin{aligned}\tilde{\phi}(\alpha(t)) = \Big(& -\sqrt{\tfrac{1}{2}+\tfrac{t}{4}} + \sqrt{\tfrac{1}{2}-\tfrac{t}{4}}\, i,\ \sqrt{\tfrac{1}{2}+\tfrac{t}{4}} + \sqrt{\tfrac{1}{2}-\tfrac{t}{4}}\, i,\\ & \sqrt{\tfrac{1}{2}-\tfrac{t+1}{4}} + \sqrt{\tfrac{1}{2}+\tfrac{t+1}{4}}\, i,\ \sqrt{\tfrac{1}{2}-\tfrac{t+1}{4}} - \sqrt{\tfrac{1}{2}+\tfrac{t+1}{4}}\, i\Big).\end{aligned} \tag{42}$$

In particular,

$$\phi(s) = \Big(-1,\ -1,\ \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\ \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\Big)$$
$$\tilde{\phi}(s) = \Big(i,\ i,\ \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i,\ \tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\Big). \tag{43}$$

Proposition 6.2. *Let $b, b' \in \widetilde{B}$, $g, g' \in V$ and $\epsilon, \epsilon' \in E$. Then*

$$\psi_g^{\epsilon}(b) = \psi_{g'}^{\epsilon'}(b') \tag{44}$$

if and only if at least one of the following holds:

(a) $b = b'$, $g = g'$ *and* $\epsilon = \epsilon'$
(b) $b = b' = q$ *and* $\epsilon = \epsilon'$
(c) $b = b' \in \partial_\ell \widetilde{B}$, $g^{-1}g' = t$ *and* $\epsilon = \epsilon'$
(d) $b = b' \in \partial_r \widetilde{B}$, $g^{-1}g' = u$ *and* $\epsilon = \epsilon'$
(e) $b = x_0$, $b' = y_0$, $(g, g') \in \{(e,t),(u,v)\}$ *and* $\epsilon\epsilon' = (-1,1,1,1)$
(f) $b = x_0$, $b' = y_0$, $(g, g') \in \{(t,e),(v,u)\}$ *and* $\epsilon\epsilon' = (1,-1,1,1)$
(g) $b = y_0$, $b' = x_0$, $(g, g') \in \{(t,e),(v,u)\}$ *and* $\epsilon\epsilon' = (-1,1,1,1)$
(h) $b = y_0$, $b' = x_0$, $(g, g') \in \{(e,t),(u,v)\}$ *and* $\epsilon\epsilon' = (1,-1,1,1)$
(i) $b = x_{-1}$, $b' = y_{-1}$, $(g, g') \in \{(e,u),(t,v)\}$ *and* $\epsilon\epsilon' = (1,1,-1,1)$
(j) $b = x_{-1}$, $b' = y_{-1}$, $(g, g') \in \{(u,e),(v,t)\}$ *and* $\epsilon\epsilon' = (1,1,1,-1)$
(k) $b = y_{-1}$, $b' = x_{-1}$, $(g, g') \in \{(u,e),(v,t)\}$ *and* $\epsilon\epsilon' = (1,1,-1,1)$
(l) $b = y_{-1}$, $b' = x_{-1}$, $(g, g') \in \{(e,u),(t,v)\}$ *and* $\epsilon\epsilon' = (1,1,1,-1)$
(m) $b = b' \in I_\ell$, $g^{-1}g' = v$ *and* $\epsilon\epsilon' = (-1,-1,1,1)$
(n) $b = b' \in I_m$, $g^{-1}g' = v$ *and* $\epsilon\epsilon' = (-1,-1,-1,-1)$
(o) $b = b' \in I_r$, $g^{-1}g' = v$ *and* $\epsilon\epsilon' = (1,1,-1,-1)$
(p) $b = b' = s$, $g^{-1}g' = u$ *and* $\epsilon\epsilon' = (-1,-1,1,1)$
(q) $b = b' = r$, $g^{-1}g' = t$ *and* $\epsilon\epsilon' = (1,1,-1,-1)$

(r) $b=\tau_0(e^{i\theta})$, $b'=\tau_0(e^{i(\pi-\theta)})$ *for some* $0\le\theta\le\pi$, $(g,g')\in\{(e,u),(u,e)\}$ *and* $\epsilon\epsilon'=(-1,1,-1,-1)$
(s) $b=\tau_0(e^{i\theta})$, $b'=\tau_0(e^{i(\pi-\theta)})$ *for some* $0\le\theta\le\pi$, $(g,g')\in\{(t,v),(v,t)\}$ *and* $\epsilon\epsilon'=(1,-1,-1,-1)$
(t) $b=\tau_{-1}(e^{i\theta})$, $b'=\tau_{-1}(e^{i(\pi-\theta)})$ *for some* $0\le\theta\le\pi$, $(g,g')\in\{(e,t),(t,e)\}$ *and* $\epsilon\epsilon'=(-1,-1,1,-1)$
(u) $b=\tau_{-1}(e^{i\theta})$, $b'=\tau_{-1}(e^{i(\pi-\theta)})$ *for some* $0\le\theta\le\pi$, $(g,g')\in\{(u,v),(v,u)\}$ *and* $\epsilon\epsilon'=(-1,-1,-1,1)$.

Proof. We will use the notation $\tilde\phi(b)=(z_1(b),z_2(b),z_3(b),z_4(b))$. The equality (44) holds if and only if either

$$\tilde\phi_g^{\epsilon}(b)=\tilde\phi_{g'}^{\epsilon'}(b') \tag{45}$$

or

$$\tilde\phi_g^{\epsilon}(b)=\overline{\tilde\phi_{g'}^{\epsilon'}(b')} \tag{46}$$

holds.

Suppose (45) holds. Then squaring yields $\phi_g(b)=\phi_{g'}(b')$, and by Lemma 6.1, at least one of conditions (i)–(vi) listed there holds. Clearly (i)$\Rightarrow$(a) and (ii)$\Rightarrow$(b). We have $z_1(b)=z_2(b)$ for all $b\in\partial_\ell\widetilde B$, by continuity and since it holds at $b=q$, and thus (iii)$\Rightarrow$(c). Similarly, since $z_3(b)=z_4(b)$ for all $b\in\partial_r\widetilde B$, we get (iv)$\Rightarrow$(d). Conversely, it is clear that any of (a)–(d) imply (45). Suppose (v) holds. Since $b,b'\in\widetilde B$, we need $\zeta=\pm1$, i.e. either $b=x_0$ and $b'=y_0$ or $b=y_0$ and $b'=x_0$. Examining (35) and (37), we obtain that one of (e)–(h) holds, and conversely any of (e)–(h) implies (45). Similarly, (vi) and (45) together are equivalent to (i)–(l).

Now suppose (46) holds. Squaring and using (29), we get

$$\phi_g(b)=\overline{\phi_{g'}(b')}=\phi_{g'v}(\beta(b')). \tag{47}$$

From Lemma 6.1, we get that at least one of the following holds:

(i′) $b=\beta(b')$ and $g^{-1}g'=v$
(ii′) $b=\beta(b')\in\partial_\ell B\cap\partial_r B$
(iii′) $b=\beta(b')\in\partial_\ell B$ and $g^{-1}g'=u$
(iv′) $b=\beta(b')\in\partial_r B$ and $g^{-1}g'=t$
(v′) $b=\tau_0(\zeta)$ and $b'=\beta(\tau_0(-\zeta))=\tau_0(-\overline{\zeta})$ for some $\zeta\in\mathbf{T}$, and $g^{-1}g'=u$
(vi′) $b=\tau_{-1}(\zeta)$ and $b'=\beta(\tau_{-1}(-\zeta))=\tau_{-1}(-\overline{\zeta})$ for some $\zeta\in\mathbf{T}$, and $g^{-1}g'=t$.

Since $b,b'\in\widetilde B$, in cases (i′)–(iv′), we have $b=\beta(b')$, so $b\in\widetilde B\cap\beta(\widetilde B)=I_\ell\cup I_m\cup I_r$. But β restricts to the identity map on $\widetilde B\cap\beta(\widetilde B)$, and we conclude $b=b'$. In particular condition (ii′) cannot hold.

Suppose (i′) holds. Then (46) becomes

$$\tilde\phi_g^{\epsilon}(b)=\overline{\tilde\phi_{gv}^{\epsilon'}(b)}. \tag{48}$$

If $b\in I_\ell$, then examining (41), (42) and (43), we conclude that (48) holds if and only if $\epsilon\epsilon'=(-1,-1,1,1)$; this corresponds to condition (m). If $b\in I_m$, then examining (33), (34) and (35), we find that (48) holds if and only if $\epsilon\epsilon'=(-1,-1,-1,-1)$; this

corresponds to condition (n). If $b \in I_r$, then examining (37), (38) and (39), we find that (48) holds if and only if $\epsilon\epsilon' = (1, 1, -1, -1)$; this corresponds to condition (o).

Suppose (iii′) holds. Then $b = b' = s$. Using (43), we conclude that (48) holds if and only if $\epsilon\epsilon' = (-1, -1, 1, 1)$; this corresponds to condition (p).

Suppose (iv′) holds. Then $b = b' = r$. Using (39), we conclude that (48) holds if and only if $\epsilon\epsilon' = (1, 1, -1, -1)$; this corresponds to condition (q).

Suppose (v′) holds, with $b = \tau_0(e^{i\theta})$, $0 \le \theta \le \pi$; we have $b' = \tau_0(e^{i(\pi-\theta)})$. Then from (36),

$$\begin{aligned}\tilde{\phi}(b) &= \left(e^{i(\frac{\theta}{2}+\frac{\pi}{4})},\, e^{i(\frac{\theta}{2}-\frac{\pi}{4})},\, \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i,\, -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\\ \overline{\tilde{\phi}(b')} &= \left(e^{i(\frac{\theta}{2}-\frac{3\pi}{4})},\, e^{i(\frac{\theta}{2}-\frac{\pi}{4})},\, \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i,\, -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right).\end{aligned}$$

Thus (48) holds if and only if (r) or (s) holds.

Similarly, if (vi′) holds, then (48) holds if and only if (t) or (u) holds. □

A consequence of Proposition 6.2 is that each of the sixty–four maps $\psi_g^\epsilon : \widetilde{B} \to \mathcal{E}/\mathbf{Z}_2$ is injective, and is thus a homeomorphism onto its image. We will use the notation $\widetilde{B}_g^\epsilon$ for the image of ψ_g^ϵ, identified with $\widetilde{B}$ via ψ_g^ϵ. The space $\mathcal{E}/\mathbf{Z}_2$ is homeomorphic to the identification space obtained by gluing together these sixty–four pieces $(\widetilde{B}_g^\epsilon)_{g\in V,\,\epsilon\in E}$ according to how they overlap in $\mathcal{E}/\mathbf{Z}_2$, namely as explicated in Proposition 6.2.

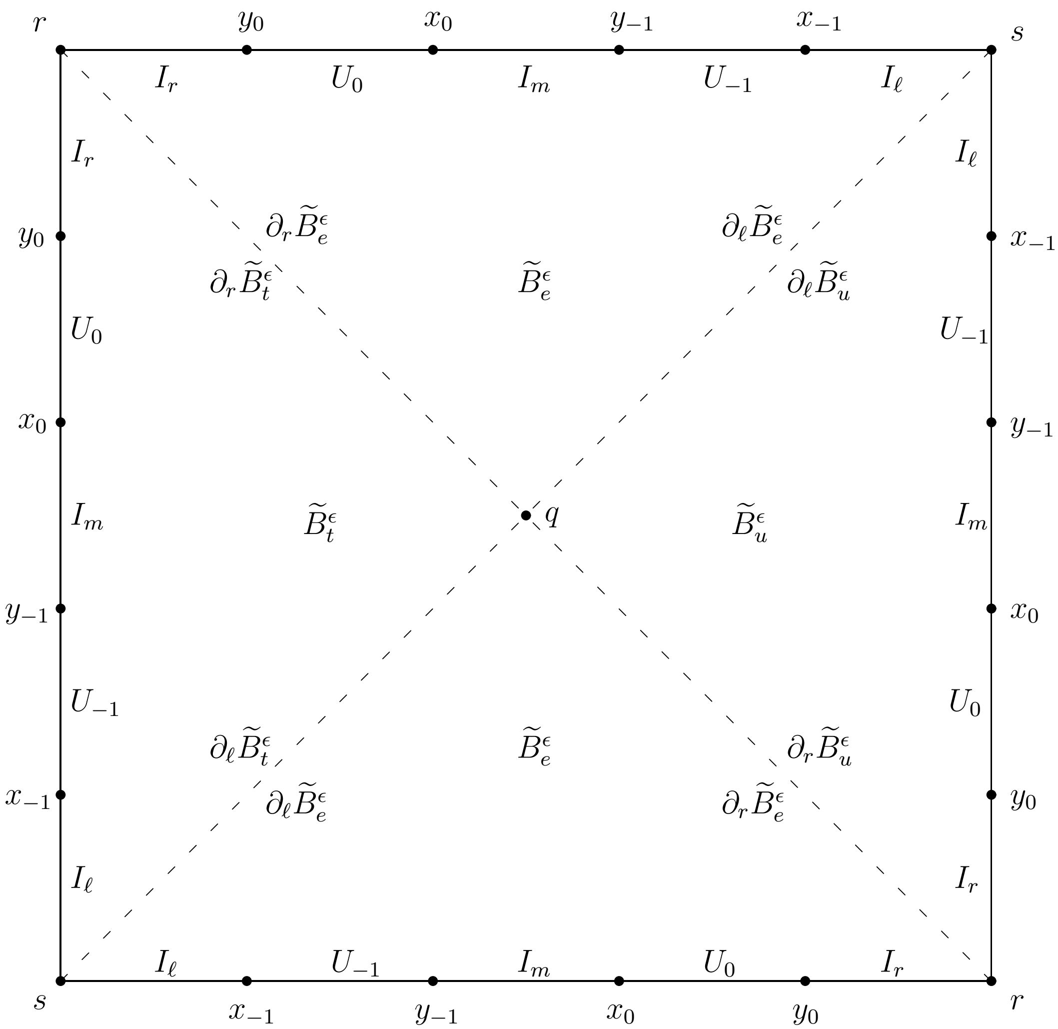
Given $\epsilon \in E$, we glue together $\widetilde{B}_e^\epsilon$, $\widetilde{B}_t^\epsilon$, $\widetilde{B}_u^\epsilon$ and $\widetilde{B}_v^\epsilon$ via (b)–(d) of Proposition 6.2 to obtain the solid square piece $\widetilde{B}^\epsilon$ depicted in Figure 7. (Note, however, that the labels on the points of the boundary of $\widetilde{B}^\epsilon$ are those retained from the picture of $\widetilde{B}$, and do not indicate that identifications of these point are made; thus, for example, the two points labeled "s" in Figure 7 are not identified with each other.) The remaining parts (e)–(u) of Proposition 6.2 are instructions for gluing the sixteen squares $(\widetilde{B}^\epsilon)_{\epsilon\in E}$ along certain of the edges and vertices, in order to obtain $\mathcal{E}/\mathbf{Z}_2$. In order to describe this space, we relabel the edges and vertices of $\widetilde{B}^\epsilon$ as shown in Figure 8. The vertices are lowercase, the edges uppercase, and we have oriented the edges as shown. We will now describe the identifications of vertices and edges that occur; all identifications of edges are orientation preserving. From (m), identify

$$\begin{aligned}\text{edges }\; A(\epsilon) &\text{ with } K((-1,-1,1,1)\epsilon)\\ J(\epsilon) &\text{ with } T((-1,-1,1,1)\epsilon)\\ \text{vertices }\; a(\epsilon) &\text{ with } k((-1,-1,1,1)\epsilon)\\ b(\epsilon) &\text{ with } \ell((-1,-1,1,1)\epsilon)\\ j(\epsilon) &\text{ with } t((-1,-1,1,1)\epsilon).\end{aligned}$$

From (n), identify

$$\begin{aligned}
\text{edges}\ \ C(\epsilon)\ &\text{with}\ M((-1,-1,-1,-1)\epsilon)\\
H(\epsilon)\ &\text{with}\ R((-1,-1,-1,-1)\epsilon)\\
\text{vertices}\ \ c(\epsilon)\ &\text{with}\ m((-1,-1,-1,-1)\epsilon)\\
d(\epsilon)\ &\text{with}\ n((-1,-1,-1,-1)\epsilon)\\
h(\epsilon)\ &\text{with}\ r((-1,-1,-1,-1)\epsilon)\\
i(\epsilon)\ &\text{with}\ s((-1,-1,-1,-1)\epsilon).
\end{aligned}$$

FIGURE 7. The square $\widetilde{B}^\epsilon$ assembled from $(\widetilde{B}^\epsilon_g)_{g\in V}$.

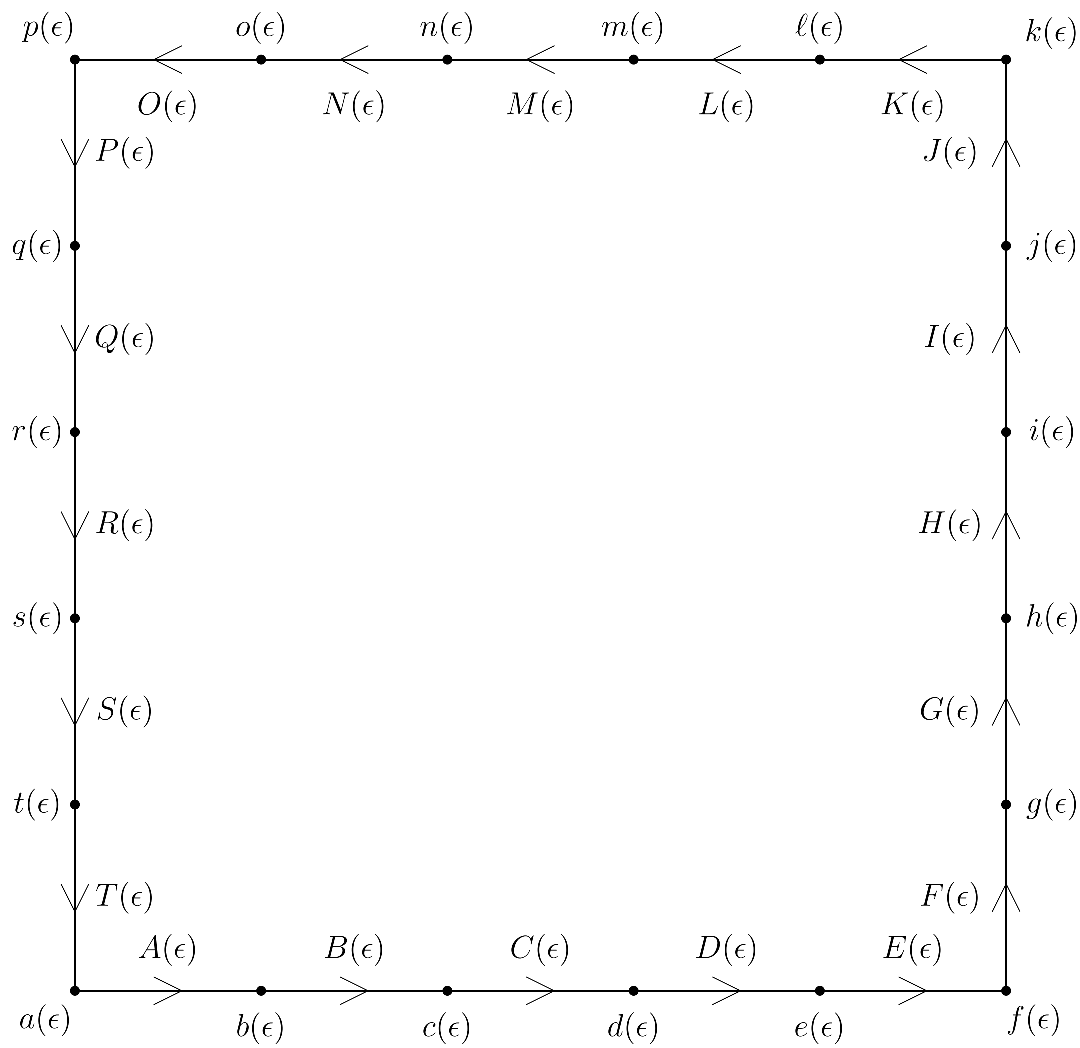


From (o), identify

$$\begin{aligned}
\text{edges } E(\epsilon) &\text{ with } O((1,1,-1,-1)\epsilon)\\
F(\epsilon) &\text{ with } P((1,1,-1,-1)\epsilon)\\
\text{vertices } \ e(\epsilon) &\text{ with } o((1,1,-1,-1)\epsilon)\\
f(\epsilon) &\text{ with } p((1,1,-1,-1)\epsilon)\\
g(\epsilon) &\text{ with } q((1,1,-1,-1)\epsilon).
\end{aligned}$$

Part (p) identifies vertices $a(\epsilon)$ with $k((-1,-1,1,1)\epsilon)$, which has already been done. Part (q) identifies vertices $f(\epsilon)$ with $p((1,1,-1,-1)\epsilon)$, which has already been done.

Figure 8. $\widetilde{B}^\epsilon$ relabeled, with edges oriented.



From (r), identify

$$\begin{aligned} \text{edges } D(\epsilon) &\text{ with } G((-1,1,-1,-1)\epsilon) \\ \text{vertices } d(\epsilon) &\text{ with } g((-1,1,-1,-1)\epsilon) \\ e(\epsilon) &\text{ with } h((-1,1,-1,-1)\epsilon). \end{aligned}$$

From (s), identify

$$\begin{aligned} \text{edges } N(\epsilon) &\text{ with } Q((1,-1,-1,-1)\epsilon) \\ \text{vertices } n(\epsilon) &\text{ with } q((1,-1,-1,-1)\epsilon) \\ o(\epsilon) &\text{ with } r((1,-1,-1,-1)\epsilon). \end{aligned}$$

From (t), identify

$$\begin{aligned} \text{edges } B(\epsilon) &\text{ with } S((-1,-1,1,-1)\epsilon) \\ \text{vertices } b(\epsilon) &\text{ with } s((-1,-1,1,-1)\epsilon) \\ c(\epsilon) &\text{ with } t((-1,-1,1,-1)\epsilon). \end{aligned}$$

From (u), identify

$$\begin{aligned} \text{edges } I(\epsilon) &\text{ with } L((-1,-1,-1,1)\epsilon) \\ \text{vertices } i(\epsilon) &\text{ with } \ell((-1,-1,-1,1)\epsilon) \\ j(\epsilon) &\text{ with } m((-1,-1,-1,1)\epsilon). \end{aligned}$$

The remaining parts of Proposition 6.2 make identifications that have already been done. Namely, part (e) identifies vertices $d(\epsilon)$ with $q((-1,1,1,1)\epsilon)$ and $h(\epsilon)$ with $o((-1,1,1,1)\epsilon)$; part (f) identifies vertices $r(\epsilon)$ with $e((1,-1,1,1)\epsilon)$ and $n(\epsilon)$ with $g((1,-1,1,1)\epsilon)$; parts (g) and (h) simply repeat parts (e) and (f); part (i) identifies vertices $b(\epsilon)$ with $i((1,1,-1,1)\epsilon)$ and $t(\epsilon)$ with $m((1,1,-1,1)\epsilon)$; part (j) identifies vertices $j(\epsilon)$ with $c((1,1,1,-1)\epsilon)$ and $\ell(\epsilon)$ with $s((1,1,1,-1)\epsilon)$; parts (k) and (l) repeat parts (i) and (j).

Therefore, the identification space, which is homeomorphic to $\mathcal{E}/\mathbf{Z}_2$, and thus to $\mathcal{G}^{\mathbf{R}}_{5,2}$, is a 2–complex having sixteen 2–cells, namely the squares $(\widetilde{B}^\epsilon)_{\epsilon\in E}$, one hundred and sixty edges,

$$\bigcup_{\epsilon\in E}\{\overline{A}(\epsilon),\overline{B}(\epsilon),\overline{C}(\epsilon),\overline{D}(\epsilon),\overline{E}(\epsilon),\overline{F}(\epsilon),\overline{H}(\epsilon),\overline{I}(\epsilon),\overline{J}(\epsilon),\overline{N}(\epsilon)\},$$

and ninety–six vertices,

$$\bigcup_{\epsilon\in E}\{\overline{a}(\epsilon),\overline{b}(\epsilon),\overline{c}(\epsilon),\overline{d}(\epsilon),\overline{e}(\epsilon),\overline{f}(\epsilon)\},$$

where

$$\begin{aligned}
\overline{A}(\epsilon) &= \{A(\epsilon), K((-1,-1,1,1)\epsilon)\}\\
\overline{B}(\epsilon) &= \{B(\epsilon), S((-1,-1,1,-1)\epsilon)\}\\
\overline{C}(\epsilon) &= \{C(\epsilon), M((-1,-1,-1,-1)\epsilon)\}\\
\overline{D}(\epsilon) &= \{D(\epsilon), G((-1,1,-1,-1)\epsilon)\}\\
\overline{E}(\epsilon) &= \{E(\epsilon), O((1,1,-1,-1)\epsilon)\}\\
\overline{F}(\epsilon) &= \{F(\epsilon), P((1,1,-1,-1)\epsilon)\}\\
\overline{H}(\epsilon) &= \{H(\epsilon), R((-1,-1,-1,-1)\epsilon)\}\\
\overline{I}(\epsilon) &= \{I(\epsilon), L((-1,-1,-1,1)\epsilon)\}\\
\overline{J}(\epsilon) &= \{J(\epsilon), T((-1,-1,1,1)\epsilon)\}\\
\overline{N}(\epsilon) &= \{N(\epsilon), Q((1,-1,-1,-1)\epsilon)\}
\end{aligned}$$

and where

$$\begin{aligned}
\overline{a}(\epsilon) &= \{a(\epsilon), k((-1,-1,1,1)\epsilon)\}\\
\overline{b}(\epsilon) &= \{b(\epsilon), i((1,1,-1,1)\epsilon), \ell((-1,-1,1,1)\epsilon), s((-1,-1,1,-1)\epsilon)\}\\
\overline{c}(\epsilon) &= \{c(\epsilon), j((1,1,1,-1)\epsilon), m((-1,-1,-1,-1)\epsilon), t((-1,-1,1,-1)\epsilon)\}\\
\overline{d}(\epsilon) &= \{d(\epsilon), g((-1,1,-1,-1)\epsilon), n((-1,-1,-1,-1)\epsilon), q((-1,1,1,1)\epsilon)\}\\
\overline{e}(\epsilon) &= \{e(\epsilon), h((-1,1,-1,-1)\epsilon), o((1,1,-1,-1)\epsilon), r((1,-1,1,1)\epsilon)\}\\
\overline{f}(\epsilon) &= \{f(\epsilon), p((1,1,-1,-1)\epsilon)\}.
\end{aligned}$$

From Theorem 4.3, $\mathcal{G}^{\mathbf{R}}_{5,2}$ is a closed, orientable surface. From the above description, the Euler characteristic of $\mathcal{G}^{\mathbf{R}}_{5,2}$ is $16 - 160 + 96 = -48$.

Theorem 6.3. *The space $\mathcal{G}^{\mathbf{R}}_{5,2}$ is homeomorphic to the closed, orientable surface of genus* 25.

7. Connectedness

Since $\mathcal{G}^{\mathbf{E}}_{k,n}$ and $\mathcal{F}^{\mathbf{E}}_{k,n}$ are real algebraic sets, each has only finitely many connected components, by the classical result of Whitney [11]. We already saw, in §3, that $\mathcal{G}^{\mathbf{C}}_{n+1,1}$ and thus $\mathcal{G}^{\mathbf{C}}_{n+1,n}$ are connected, while $\mathcal{G}^{\mathbf{R}}_{n+1,1}$ and $\mathcal{G}^{\mathbf{R}}_{n+1,n}$ are disconnected, for $n \geq 1$.

Recall from Corollary 2.6 we have the locally trivial fiber bundle $\pi^{\mathbf{E}}_{k,n} : \mathcal{F}^{\mathbf{E}}_{k,n} \to \mathcal{G}^{\mathbf{E}}_{k,n}$ and $\mathcal{G}^{\mathbf{E}}_{k,n}$ is thereby identified with the orbit space $\mathcal{F}^{\mathbf{E}}_{k,n}/\mathcal{O}^{\mathbf{E}}_{n}$. Moreover, from Corollary 2.7, we have the homeomorphism $\gamma_{k,n} : \mathcal{G}^{\mathbf{E}}_{k,n} \to \mathcal{G}^{\mathbf{E}}_{k,k-n}$.

In the complex case, since the fibers $\mathcal{O}^{\mathbf{C}}_{n}$ are connected, the following result is obvious.

Proposition 7.1. *Let $k, n \in \mathbf{N}$ with $k > n$. Let C be a connected component of $\mathcal{G}^{\mathbf{C}}_{k,n}$. Then $(\pi^{\mathbf{C}}_{k,n})^{-1}(C)$ is connected. Thus, $\mathcal{F}^{\mathbf{C}}_{k,n}$ and $\mathcal{G}^{\mathbf{C}}_{k,n}$ have the same number of*

connected components; in particular, $\mathcal{F}^{\mathbf{C}}_{k,n}$ is connected if and only if $\mathcal{G}^{\mathbf{C}}_{k,n}$ is connected. Furthermore, $\mathcal{F}^{\mathbf{C}}_{k,n}$ is connected if and only if $\mathcal{F}^{\mathbf{C}}_{k,k-n}$ is connected.

The real case, however, is somewhat more interesting.

Proposition 7.2. *Let $k,n \in \mathbf{N}$ with $k > n$. Let C be a connected component of $\mathcal{G}^{\mathbf{R}}_{k,n}$. Then $(\pi^{\mathbf{R}}_{k,n})^{-1}(C)$ is either connected or has exactly two connected components. Furthermore, $(\pi^{\mathbf{R}}_{k,n})^{-1}(C)$ and $(\pi^{\mathbf{R}}_{k,k-n})^{-1}(\gamma_{k,n}(C))$ have the same number of connected components.*

Proof. Let $\mathcal{F}^{\mathbf{R}}_{k,n}/\mathcal{SO}_n$ denote the orbit space of the restriction of the action of $\mathcal{O}^{\mathbf{R}}_n$ on $\mathcal{F}^{\mathbf{R}}_{k,n}$ to the special orthogonal group $\mathcal{SO}_n \subset \mathcal{O}^{\mathbf{R}}_n$. Since $\pi^{\mathbf{R}}_{k,n}$ is a locally trivial fiber bundle, we see that the quotient $q : \mathcal{F}^{\mathbf{R}}_{k,n}/\mathcal{SO}_n \to \mathcal{G}^{\mathbf{R}}_{k,n}$ is a two–fold covering projection, and that $(\pi^{\mathbf{R}}_{k,n})^{-1}(C)$ and $q^{-1}(C)$ have the same number of connected components; in particular this number is either one or two.

Since $\mathcal{G}^{\mathbf{R}}_{k,n}$ is a real algebraic variety, by Whitney's results [11] (see also §4) it is locally path connected and thus C is path connected. If $(\pi^{\mathbf{R}}_{k,n})^{-1}(C)$ is connected, then there is a closed path $\tau : [0,1] \to C$ with the property that if $\tilde{\tau} : [0,1] \to \mathcal{F}^{\mathbf{R}}_{k,n}$ is a lifting, then letting $U \in \mathcal{O}_n$ be such that $\tilde{\tau}(1) = U\tilde{\tau}(0)$, we have $\det(U) = -1$. Let $\tilde{\tau}' : [0,1] \to \mathcal{F}^{\mathbf{R}}_{k,k-n}$ be a lifting of $\gamma_{k,n} \circ \tau : [0,1] \to \mathcal{G}^{\mathbf{R}}_{k,k-n}$ and let $U' \in \mathcal{O}_{k-n}$ be such that $\tilde{\tau}'(1) = U'\tilde{\tau}'(0)$. We will show $\det(U') = -1$, which will imply $(\pi^{\mathbf{R}}_{k,k-n})^{-1}(C)$ is connected and will thus finish the proof. Let $P(t) = \frac{n}{k}\tau(t)$. Then, *cf* (3),

$$V(t) = \sqrt{\frac{n}{k}} W^*_{n,k}\tilde{\tau}(t) \in M_k(\mathbf{R}), \quad (t \in [0,1])$$

is a continuous path of partial isometries satisfying

$$V(t)^*V(t) = P(t), \qquad V(t)V(t)^* = \mathrm{diag}(\underbrace{1,\ldots,1}_{n},0,\ldots,0), \qquad (t \in [0,1])$$

and $V(1)V(0)^* = \left(\begin{smallmatrix} U & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Similarly,

$$V'(t) = \sqrt{\frac{k-n}{n}} \begin{pmatrix} 0_{n,k-n} \\ I_{k-n} \end{pmatrix} \tilde{\tau}'(t) \in M_k(\mathbf{R}), \quad (t \in [0,1])$$

is a continuous path of partial isometries satisfying

$$V'(t)^*V'(t) = I_k - P(t), \qquad V'(t)V'(t)^* = \mathrm{diag}(0,\ldots,0,\underbrace{1,\ldots,1}_{k-n}), \qquad (t \in [0,1])$$

and $V'(1)V'(0)^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & U' \end{smallmatrix}\right)$. Therefore, $(V(t)+V'(t))(V(0)+V'(0))^*$, $0 \le t \le 1$, is a continuous path in $\mathcal{O}^{\mathbf{R}}_k$ starting at I_k and ending at $\left(\begin{smallmatrix} U & 0 \\ 0 & U' \end{smallmatrix}\right)$. This implies $\det(U) = \det(U')$. □

Corollary 7.3. *Let $k,n \in \mathbf{N}$, $k > n$. Then $\mathcal{F}^{\mathbf{R}}_{k,n}$ is connected if and only if $\mathcal{F}^{\mathbf{R}}_{k,k-n}$ is connected.*

Theorem 7.4. *Let $k \in \mathbf{N}$, $k \ge 4$. Then $\mathcal{F}^{\mathbf{R}}_{k,2}$ is connected.*

Proof. By Corollary 5.2, $\mathcal{F}^{\mathbf{R}}_{k,2}$ is homeomorphic to

$$\widetilde{\mathcal{F}}_{k,2} = \{(z_1, \dots, z_k) \in \mathbf{T}^k \mid \sum_{j=1}^{k} z_j^2 = 0\}.$$

Let

$$\mathcal{C} = \{(w_1, \dots, w_k) \in \mathbf{T}^k \mid \sum_{j=1}^{k} w_j = 0\}$$

and let $p : \widetilde{\mathcal{F}}_{k,2} \to \mathcal{C}$ be $p((z_1, \dots, z_k)) = (z_1^2, \dots, z_k^2)$. Then p is a 2^k–fold covering projection.

Thus $\mathcal{C}$ is the space of all chains in $\mathbf{R}^2$, starting and ending at 0 and having links of uniform length 1. It is known (see, for example, [9, Thm. 3.1]) that the space $\mathcal{C}$ is path connected. Thus, given any $c \in \mathcal{C}$, there is a path from c to some chosen element $s \in \mathcal{C}$, that is said to be in *standard form* and is described below. Of course, given a path $\gamma : [0,1] \to \mathcal{C}$ and given $a \in p^{-1}(\gamma(0))$, there is a (unique) path $\gamma' : [0,1] \to \widetilde{\mathcal{F}}_{k,2}$ such that $p \circ \gamma' = \gamma$ and $\gamma'(0) = a$. Therefore, in order to show connectedness of $\widetilde{\mathcal{F}}_{k,2}$, it will suffice to specify a particular element $b \in p^{-1}(s)$ and to exhibit a path in $\widetilde{\mathcal{F}}_{k,2}$ from each element of $p^{-1}(s)$ to b.

Case I: $k = 4$. We take as standard element $s = (1, -1, 1, -1) \in \mathcal{C}$. Then

$$p^{-1}(s) = \{(\epsilon_1, \epsilon_2 i, \epsilon_3, \epsilon_4 i) \mid \epsilon_j \in \{\pm 1\}\},$$

and we select $b = (1, i, 1, i)$. It is not difficult to construct paths in $\widetilde{\mathcal{F}}_{4,2}$ from all elements of $p^{-1}(s)$ to b. This is somewhat tedious and is left to the reader.

Alternatively, it is proved in §5 that $\mathcal{G}^{\mathbf{R}}_{4,2}$ is connected; the points b and $a = (1, -i, 1, -i)$ in $\widetilde{\mathcal{F}}_{4,2}$ belong to the same $\mathcal{O}^{\mathbf{R}}_2$–orbit but differ by a matrix in $\mathcal{O}^{\mathbf{R}}_2$ of determinant -1; therefore, by the technique of the proof of Proposition 7.2, in order to show that $\widetilde{\mathcal{F}}_{4,2}$ is connected, it will suffice to find a path in $\widetilde{\mathcal{F}}_{4,2}$ from b to a. Starting at b the path

$$(e^{i\theta}, ie^{i\theta}, 1, i), \qquad 0 \le \theta \le \pi$$

takes us to $(-1, -i, 1, i)$; then the path

$$(e^{i\theta}, -i, 1, ie^{i\theta}), \qquad 0 \le \theta \le \pi,$$

takes us to $(1, -i, 1, -i) = a$.

Case II: k even, $k \ge 6$. Take as standard element $s = (1, -1, 1, -1, \dots, 1, -1) \in \mathcal{C}$, so

$$p^{-1}(s) = \{(\epsilon_1, \epsilon_2 i, \epsilon_3, \epsilon_4 i, \dots, \epsilon_{k-1}, \epsilon_k i) \mid \epsilon_j \in \{\pm 1\}\}.$$

Take $b = (1, i, \dots, 1, i)$ and, given $a = (\epsilon_1, \epsilon_2 i, \dots, \epsilon_{k-1}, \epsilon_k i) \in p^{-1}(s)$, construct a path in $\widetilde{\mathcal{F}}_{k,2}$ from a to b as follows. We have $(\epsilon_1, \epsilon_2 i, \epsilon_3, \epsilon_4 i) \in \widetilde{\mathcal{F}}_{4,2}$, and by Case I, $\widetilde{\mathcal{F}}_{4,2}$ is connected, so there is a path in $\widetilde{\mathcal{F}}_{4,2}$ from $(\epsilon_1, \epsilon_2 i, \epsilon_3 \epsilon_4 i)$ to $(1, i, 1, i)$. Keeping the remaining $k - 4$ elements constant, this yields a path in $\widetilde{\mathcal{F}}_{k,2}$ from a to

$$(1, i, 1, i, \epsilon_5, \epsilon_6 i, \dots, \epsilon_{k-1}, \epsilon_{k-1} i). \tag{49}$$

Now taking a path in $\widetilde{\mathcal{F}}_{4,2}$ from $(1, i, \epsilon_5, \epsilon_6 i)$ to $(1, i, 1, i)$ yields a path in $\widetilde{\mathcal{F}}_{k,2}$ from the point in (49) to

$$(1, i, 1, i, 1, i, \epsilon_7, \epsilon_8 i, \ldots, \epsilon_{k-1}, \epsilon_{k-1} i).$$

Continuing in this manner, we construct a path in $\widetilde{\mathcal{F}}_{k,2}$ from a to b.

Case III: $k = 5$. Take as standard element

$$s = (-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i, -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i, 1, 1, -1) \in \mathcal{C}.$$

Then

$$p^{-1}(s) = \{(\epsilon_1(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i), \epsilon_2(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i), \epsilon_3, \epsilon_4, \epsilon_5 i) \mid \epsilon_j \in \{\pm 1\}\}$$

and we select

$$b = (\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i, \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i, 1, 1, i) = (e^{\pi i/3}, e^{-\pi i/3}, 1, 1, i).$$

It is again routine, though tedious, to construct paths in $\widetilde{\mathcal{F}}_{5,2}$ from all the elements of $p^{-1}(s)$ to b.

Alternatively, arguing as in Case I above, using the result from §6 that $\mathcal{G}^{\mathbf{R}}_{5,2}$ is connected, it will suffice to construct a path in $\widetilde{\mathcal{F}}_{5,2}$ from b to

$$a = (\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i, \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i, 1, 1, -i) = (e^{-\pi i/3}, e^{\pi i/3}, 1, 1, -i).$$

The path

$$(e^{\pi i/3}, e^{-\pi i/3}, e^{i\theta}, 1, ie^{i\theta}), \qquad 0 \le \theta \le \pi/3$$

takes us from b to $(e^{\pi i/3}, e^{-\pi i/3}, e^{\pi i/3}, 1, e^{5\pi i/6})$; the path

$$(e^{i(\theta + \pi/3)}, e^{-\pi i/3}, e^{\pi i/3}, 1, e^{i(\theta + 5\pi/6)}), \qquad 0 \le \theta \le 4\pi/3$$

takes us to $(e^{-\pi i/3}, e^{-\pi i/3}, e^{\pi i/3}, 1, e^{\pi i/6})$; the path

$$(e^{-\pi i/3}, e^{i(\theta - \pi/3)}, e^{\pi i/3}, 1, e^{i(\theta + \pi/6)}), \qquad 0 \le \theta \le \pi/6$$

takes us to $(e^{-\pi i/3}, e^{-\pi i/6}, e^{\pi i/3}, 1, e^{\pi i/3})$; the path

$$(e^{-\pi i/3}, e^{i(\theta - \pi/6)}, e^{i(\theta + \pi/3)}, 1, e^{\pi i/3}), \qquad 0 \le \theta \le \pi/2$$

takes us to $(e^{-\pi i/3}, e^{\pi i/3}, e^{5\pi i/6}, 1, e^{\pi i/3})$; the path

$$(e^{-\pi i/3}, e^{\pi i/3}, e^{i(\theta + 5\pi/6)}, 1, e^{i(\theta + \pi/3)}), \qquad 0 \le \theta \le 7\pi/6$$

takes us to a.

Case IV: k odd, $k \ge 5$. Take as standard element

$$s = (-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i, -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i, 1, 1, -1, 1, -1, \ldots, 1, -1) \in \mathcal{C}.$$

and select

$$b = (\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i, \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i, 1, 1, i) = (e^{\pi i/3}, e^{-\pi i/3}, 1, 1, i, 1, i, \ldots, 1, i) \in p^{-1}(s).$$

We must find a path in $\widetilde{\mathcal{F}}_{k,2}$ from an arbitrary element

$$c = (\epsilon_1(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i), \epsilon_2(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i), \epsilon_3, \epsilon_4, \epsilon_5 i, \ldots, \epsilon_{k-1}, \epsilon_k i), \qquad \epsilon_j \in \{\pm 1\}$$

of $p^{-1}(s)$ to b. Similarly to in Case II above, using paths in $\widetilde{\mathcal{F}}_{5,2}$, which by Case III we know to be connected, we construct a path from c to b passing through the points

$$\begin{aligned}
&(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i, \tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i, 1, 1, i, \epsilon_6, \epsilon_7 i, \epsilon_8, \epsilon_9 i, \ldots, \ldots, \epsilon_{k-1}, \epsilon_k i) \\
&(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i, \tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i, 1, 1, i, 1, i, \epsilon_8, \epsilon_9 i, \ldots, \ldots, \epsilon_{k-1}, \epsilon_k i) \\
&\vdots \\
&(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i, \tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i, 1, 1, i, 1, i, \ldots, 1, i, \epsilon_{k-1}, \epsilon_k i)
\end{aligned}$$

in succession. □

Now from Corollary 7.3 we immediately have the following:

Corollary 7.5. *Let $n \in \mathbf{N}$, $n \geq 2$. Then $\mathcal{F}^{\mathbf{R}}_{n+2,n}$ is connected.*

We take Theorem 7.4 and Corollary 7.5 to be strong indications:

Conjecture 7.6. *Let $k, n \in \mathbf{N}$, with $n \geq 2$ and $k \geq n+2$. Then $\mathcal{F}^{\mathbf{R}}_{k,n}$ is connected.*

We also conjecture connectedness in the complex case:

Conjecture 7.7. *Let $k, n \in \mathbf{N}$, with $n \geq 1$ and $k > n$. Then $\mathcal{F}^{\mathbf{C}}_{k,n}$ is connected.*

References

[1] J.J. Benedetto and M. Fickus, *Finite normalized tight frames,* Adv. Comput. Math. **18** (2003), 357-385.

[2] P.G. Casazza and J.K. Kovačević, *Equal–norm tight frames with erasures,* Adv. Comput. Math. **18** (2003), 387-430.

[3] P. Casazza, J. Kovačević, M. Leon and J.C. Treman, *Custom built tight frames,* preprint.

[4] P.G. Casazza and M. Leon, *Existence and construction of finite frames with a given frame operator,* preprint, (2002).

[5] O. Christensen, *An Introduction to Frames and Riesz Bases,* Birhäuser, Boston, 2003.

[6] K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower and E. Weber, *Ellipsoidal tight frames and projection decompositions of operators,* Illinois J. Math., to appear.

[7] V.K. Goyal, J. Kovačević and J.A. Kelner, *Quantized frame expansions with erasures,* Appl. Comput. Harmon. Anal. **10** (2001), 203-233.

[8] R.B. Holmes and V.I. Paulsen, *Optimal frames for erasures,* preprint.

[9] W.J. Lenhart and S.H. Whitesides, *Reconfiguring closed polygonal chains in Euclidean d–space,* Discrete Comput. Geom. **13** (1995), 123-140.

[10] T. Strohmer and R. Heath, *Grassmannian frames with applications to coding and communication,* preprint (2003), arXiv.math.FA/0301135.

[11] H. Whitney, *Elementary structure of real algebraic varieties,* Ann. of Math. **66** (1957), 454-556.

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